CUBES IN PRODUCTS OF TERMS IN ARITHMETIC PROGRESSION

L. HAJDU, SZ. TENGELY, R. TIJDEMAN

Abstract. Euler proved that the product of four positive integers in arithmetic progression is not a square. Győry, using a result of Darmon and Merel, showed that the product of three coprime positive integers in arithmetic progression cannot be an l-th power for $l \geq 3$. There is an extensive literature on longer arithmetic progressions such that the product of the terms is an (almost) power. In this paper we extend the range of k's such that the product of k coprime integers in arithmetic progression cannot be a cube when $2 < k < 39$. We prove a similar result for almost cubes.

1. Introduction

In this paper we consider the problem of almost cubes in arithmetic progressions. This problem is closely related to the Diophantine equation

$$n(n+d)\dots(n+(k-1)d) = by^l \tag{1}$$

in positive integers n, d, k, b, y, l with $l \geq 2$, $k \geq 3$, $\gcd(n, d) = 1$, $P(b) \leq k$, where for $u \in \mathbb{Z}$ with $|u| > 1$, $P(u)$ denotes the greatest prime factor of u, and $P(\pm 1) = 1$.

This equation has a long history, with an extensive literature. We refer to the research and survey papers [3], [10], [11], [14], [16], [18], [19], [20], [23], [25], [26], [28], [29], [31], [32], [33], [34], [35], [36], [37], [38], [40], [41], the references given there, and the other papers mentioned in the introduction.

In this paper we concentrate on results where all solutions of (1) have been determined, under some assumptions for the unknowns. We start with results concerning squares, so in this paragraph we assume

2000 *Mathematics Subject Classification.* 11D41, 11D25, 11B25.

Key words and phrases. Perfect cubes, arithmetic progressions.

L. Hajdu is supported in part by the Hungarian Academy of Sciences and by the OTKA grants T48791 and K67580. Sz. Tengely is supported in part the Hungarian Academy of Sciences, by the Magyary Zoltán Higher Educational Public Foundation, and by the OTKA grant T48791.

that $l = 2$. Already Euler proved that in this case equation (1) has no solutions with $k = 4$ and $b = 1$ (see [7] pp. 440 and 635). Obláth [21] extended this result to the case $k = 5$. Erdős [8] and Rigge [22] independently proved that equation (1) has no solutions with $b = d = 1$. Saradha and Shorey [27] proved that (1) has no solutions with $b = 1$, $k \geq 4$, provided that d is a power of a prime number. Later, Laishram and Shorey [19] extended this result to the case where either $d \leq 10^{10}$, or d has at most six prime divisors. Finally, most importantly from the viewpoint of the present paper, Hirata-Kohno, Laishram, Shorey and Tijdeman [17] completely solved (1) with $3 \leq k < 110$ for $b = 1$. Combining their result with those of Tengely [39] all solutions of (1) with $3 \leq k \leq 100$, $P(b) < k$ are determined.

Now assume for this paragraph that $l \geq 3$. Erdős and Selfridge [9] proved the celebrated result that equation (1) has no solutions if $b = d = 1$. In the general case $P(b) \leq k$ but still with $d = 1$, Saradha [24] for $k \geq 4$ and Győry [12], using a result of Darmon and Merel [6], for $k = 2, 3$ proved that (1) has no solutions with $P(y) > k$. For general d, Győry [13] showed that equation (1) has no solutions with $k = 3$, provided that $P(b) \leq 2$. Later, this result has been extended to the case $k < 12$ under certain assumptions on $P(b)$, see Győry, Hajdu, Saradha [15] for $k < 6$ and Bennett, Bruin, Győry, Hajdu [1] for $k < 12$.

In this paper we consider the problem for cubes, that is equation (1) with $l = 3$. We solve equation (1) nearly up to $k = 40$. In the proofs of our results we combine the approach of [17] with results of Selmer [30] and some new ideas.

2. Notation and results

As we are interested in cubes in arithmetic progressions, we take $l = 3$ in (1). That is, we consider the Diophantine equation

$$n(n + d) \dots (n + (k - 1)d) = by^3 \tag{2}$$

in integers n, d, k, b, y where $k \geq 3$, $d > 0$, $\gcd(n, d) = 1$, $P(b) \leq k$, $n \neq 0, y \neq 0$. (Note that similarly as e.g. in [1] we allow $n < 0$, as well.)

In the standard way, by our assumptions we can write

$$n + id = a_i x_i^3 \quad (i = 0, 1, \dots, k - 1) \tag{3}$$

with $P(a_i) \leq k$, a_i is cube-free. Note that (3) also means that in fact $n + id$ $(i = 0, 1, \dots, k-1)$ is an arithmetic progression of almost cubes.

In case of $b = 1$ we prove the following result.

Theorem 2.1. *Suppose that* (n, d, k, y) *is a solution to equation* (2) *with* $b = 1$ *and* $k < 39$. *Then we have*

$$(n, d, k, y) = (-4, 3, 3, 2), (-2, 3, 3, -2), (-9, 5, 4, 6) \text{ or } (-6, 5, 4, 6).$$

We shall deduce Theorem 2.1 from the following theorem.

Theorem 2.2. *Suppose that* (n, d, k, b, y) *is a solution to equation* (2) *with* $k < 32$ *and that* $P(b) < k$ *if* $k = 3$ *or* $k \geq 13$. *Then* (n, d, k) *belongs to the following list:*

$$\begin{gathered}(n, 1, k) \text{ with } -30 \leq n \leq -4 \text{ or } 1 \leq n \leq 5,\\ (n, 2, k) \text{ with } -29 \leq n \leq -3,\\ (-10, 3, 7), (-8, 3, 7), (-8, 3, 5), (-4, 3, 5), (-4, 3, 3), (-2, 3, 3),\\ (-9, 5, 4), (-6, 5, 4), (-16, 7, 5), (-12, 7, 5).\end{gathered}$$

Note that the above statement follows from Theorem 1.1 of Bennett, Bruin, Győry, Hajdu [1] in case $k < 12$ and $P(b) \leq P_k$ with $P_3 = 2$, $P_4 = P_5 = 3$, $P_6 = P_7 = P_8 = P_9 = P_{10} = P_{11} = 5$.

3. Lemmas and auxiliary results

We need some results of Selmer [30] on cubic equations.

Lemma 3.1. *The equations*

$$\begin{aligned} &x^3 + y^3 = cz^3, \quad c \in \{1, 2, 4, 5, 10, 25, 45, 60, 100, 150, 225, 300\},\\ &ax^3 + by^3 = z^3, \ (a, b) \in \{(2, 9), (4, 9), (4, 25), (4, 45), (12, 25)\}\end{aligned}$$

have no solution in non-zero integers x, y, z.

As a lot of work will be done modulo 13, the following lemma will be very useful. Before stating it, we need to introduce a new notation. For $u, v, m \in \mathbb{Z}$, $m > 1$ by $u \overset{c}{\equiv} v \pmod{m}$ we mean that $uw^3 \equiv v \pmod{m}$ holds for some integer w with $\gcd(m, w) = 1$. We shall use this notation throughout the paper, without any further reference.

Lemma 3.2. *Let* n, d *be integers. Suppose that for five values* $i \in \{0, 1, ..., 12\}$ *we have* $n + id \overset{c}{\equiv} 1 \pmod{13}$. *Then* $13 \mid d$, *and* $n + id \overset{c}{\equiv} 1 \pmod{13}$ *for all* $i = 0, 1, \dots, 12$.

Proof. Suppose that $13 \nmid d$. Then there is an integer r such that $n \equiv rd \pmod{13}$. Consequently, $n + id \equiv (r + i)d \pmod{13}$. A simple calculation yields that the cubic residues of the numbers $(r + i)d$ ($i =$

$0, 1, \dots, 12)$ modulo 13 are given by a cyclic permutation of one of the sequences

$$0, 1, 2, 2, 4, 1, 4, 4, 1, 4, 2, 2, 1,$$
$$0, 2, 4, 4, 1, 2, 1, 1, 2, 1, 4, 4, 2,$$
$$0, 4, 1, 1, 2, 4, 2, 2, 4, 2, 1, 1, 4.$$

Thus the statement follows. $\square$

Lemma 3.3. *Let $\alpha = \sqrt[3]{2}$ and $\beta = \sqrt[3]{3}$. Put $K = \mathbb{Q}(\alpha)$ and $L = \mathbb{Q}(\beta)$. Then the only solution of the equation*

$$\mathcal{C}_1 : \quad X^3 - (\alpha+1)X^2 + (\alpha+1)X - \alpha = (-3\alpha+6)Y^3$$

in $X \in \mathbb{Q}$ and $Y \in K$ is $(X, Y) = (2, 1)$. Further, the equation

$$\mathcal{C}_2 : \quad 4X^3 - (4\beta+2)X^2 + (2\beta+1)X - \beta = (-3\beta+3)Y^3$$

has the single solution $(X, Y) = (1, 1)$ in $X \in \mathbb{Q}$ and $Y \in L$.

Proof. Using the point $(2, 1)$ we can transform the genus 1 curve $\mathcal{C}_1$ to Weierstrass form

$$E_1 : \quad y^2 + (\alpha^2+\alpha)y = x^3 + (26\alpha^2 - 5\alpha - 37).$$

We have $E_1(K) \simeq \mathbb{Z}$ as an Abelian group and $(x, y) = (-\alpha^2 - \alpha + 3, -\alpha^2 - 3\alpha + 4)$ is a non-torsion point on this curve. Applying elliptic Chabauty (cf. [4], [5]), in particular the procedure "Chabauty" of MAGMA (see [2]) with $p = 5$, we obtain that the only point on $\mathcal{C}_1$ with $X \in \mathbb{Q}$ is $(2, 1)$.

Now we turn to the second equation $\mathcal{C}_2$. We can transform this equation to an elliptic one using its point $(1, 1)$. We get

$$E_2 : \quad y^2 = x^3 + \beta^2 x^2 + \beta x + (41\beta^2 - 58\beta - 4).$$

We find that $E_2(L) \simeq \mathbb{Z}$ and $(x, y) = (4\beta - 2, -2\beta^2 + \beta + 12)$ is a non-torsion point on E_2. Applying elliptic Chabauty (as above) with $p = 11$, we get that the only point on $\mathcal{C}_2$ with $X \in \mathbb{Q}$ is $(1, 1)$. $\square$

4. Proofs

In this section we provide the proofs of our results. As Theorem 2.1 follows from Theorem 2.2 by a simple inductive argument, first we give the proof of the latter result.

Proof of Theorem 2.2. As we mentioned, for $k = 3, 4$ the statement follows from Theorem 1.1 of [1]. Observe that the statement for every

$$k \in \{6, 8, 9, 10, 12, 13, 15, 16, 17, 19, 21, 22, 23, 25, 26, 27, 28, 29, 31\}$$

is a simple consequence of the result obtained for some smaller value of k. Indeed, for any such k let p_k denote the largest prime with $p_k < k$. Observe that in case of $k \leq 13$ $P(a_0 a_1 \dots a_{p_k - 1}) \leq p_k$ holds, and for $k > 13$ we have $P(a_0 a_1 \dots a_{p_k}) < p_k + 1$. Hence, noting that we assume $P(b) \leq k$ for $3 < k \leq 11$ and $P(b) < k$ otherwise, the theorem follows inductively from the case of p_k-term products and p_k+1-term products, respectively. Hence in the sequel we deal only with the remaining values of k.

The cases $k = 5, 7$ are different from the others. In most cases a "brute force" method suffices. In the remaining cases we apply the elliptic Chabauty method (see [4], [5]).

The case $k = 5$. In this case a very simple algorithm works already. Note that in view of Theorem 1.1 of [1], by symmetry it is sufficient to assume that $5 \mid a_2 a_3$. We look at all the possible distributions of the prime factors $2, 3, 5$ of the coefficients a_i $(i = 0, \dots, 4)$ one-by-one. Using that if x is an integer, then x^3 is congruent to ± 1 or 0 both $\pmod 7$ and $\pmod 9$, almost all possibilities can be excluded. For example,

$$(a_0, a_1, a_2, a_3, a_4) = (1, 1, 1, 10, 1)$$

is impossible modulo 7, while

$$(a_0, a_1, a_2, a_3, a_4) = (1, 1, 15, 1, 1)$$

is impossible modulo 9. (Note that the first choice of the a_i cannot be excluded modulo 9, and the second one cannot be excluded modulo 7.)

In case of the remaining possibilities, taking the linear combinations of three appropriately chosen terms of the arithmetic progression on the left hand side of (2) we get all solutions by Lemma 3.1. For example,

$$(a_0, a_1, a_2, a_3, a_4) = (2, 3, 4, 5, 6)$$

obviously survives the above tests modulo 7 and modulo 9. However, in this case using the identity $4(n+d) - 3n = n + 4d$, Lemma 3.1 implies that the only corresponding solution is given by $n = 2$ and $d = 1$.

After having excluded all quintuples which do not pass the above tests we are left with the single possibility

$$(a_0, a_1, a_2, a_3, a_4) = (2, 9, 2, 5, 12).$$

Here we have

$$x_0^3 + x_2^3 = 9x_1^3 \text{ and } x_0^3 - 2x_2^3 = -6x_4^3. \tag{4}$$

Factorizing the first equation of (4), a simple consideration yields that $x_0^2 - x_0 x_2 + x_2^2 = 3u^3$ holds for some integer u. Put $K = \mathbb{Q}(\alpha)$ with $\alpha = \sqrt[3]{2}$. Note that the ring O_K of integers of K is a unique factorization domain, $\alpha-1$ is a fundamental unit and $1, \alpha, \alpha^2$ is an integral basis of K, and $3 = (\alpha-1)(\alpha+1)^3$, where $\alpha+1$ is a prime in O_K. A simple calculation shows that $x_0 - \alpha x_2$ and $x_0^2 + \alpha x_0 x_2 + \alpha^2 x_2^2$ can have only the prime divisors α and $\alpha+1$ in common. Hence checking the field norm of $x_0 - \alpha x_2$, by the second equation of (4) we get that

$$x_0 - \alpha x_2 = (\alpha-1)^\varepsilon (\alpha^2+\alpha) y^3$$

with $y \in O_K$ and $\varepsilon \in \{0,1,2\}$. Expanding the right hand side, we deduce that $\varepsilon = 0, 2$ yields $3 \mid x_0$, which is a contradiction. Thus we get that $\varepsilon = 1$, and we obtain the equation

$$(x_0 - \alpha x_2)(x_0^2 - x_0 x_2 + x_2^2) = (-3\alpha+6) z^3$$

for some $z \in O_K$. Hence after dividing both sides of this equation by x_2^3, the theorem follows from Lemma 3.3 in this case.

The case $k = 7$. In this case by similar tests as for $k = 5$, we get that the only remaining possibilities are given by

$$(a_0, a_1, a_2, a_3, a_4, a_5, a_6) = (4,5,6,7,1,9,10), (10,9,1,7,6,5,4).$$

By symmetry it is sufficient to deal with the first case. Then we have

$$x_1^3 + 8x_6^3 = 9x_5^3 \text{ and } x_6^3 - 3x_1^3 = -2x_0^3. \tag{5}$$

Factorizing the first equation of (5), just as in case of $k = 5$, a simple consideration gives that $4x_6^2 - 2x_1x_6 + x_1^2 = 3u^3$ holds for some integer u. Let $L = \mathbb{Q}(\beta)$ with $\beta = \sqrt[3]{3}$. As is well-known, the ring O_L of integers of L is a unique factorization domain, $2-\beta^2$ is a fundamental unit and $1, \beta, \beta^2$ is an integral basis of L. Further, $2 = (\beta-1)(\beta^2+\beta+1)$, where $\beta - 1$ and $\beta^2+\beta+1$ are primes in O_L, with field norms 2 and 4, respectively. A simple calculation yields that $x_6 - \beta x_1$ and $x_6^2 + \beta x_1 x_6 + \beta^2 x_1^2$ are relatively prime in O_L. Moreover, as $\gcd(n,d) = 1$ and x_4 is even, x_0 should be odd. Hence as the field norm of $\beta^2+\beta+1$ is 4, checking the field norm of $x_6 - \beta x_1$, the second equation of (5) yields

$$x_6 - \beta x_1 = (2-\beta^2)^\varepsilon (1-\beta) y^3$$

for some $y \in O_L$ and $\varepsilon \in \{0,1,2\}$. Expanding the right hand side, a simple computation shows that $\varepsilon = 1, 2$ yields $3 \mid x_6$, which is a contradiction. Thus we get that $\varepsilon = 0$, and we obtain the equation

$$(x_6 - \beta x_1)(4x_6^2 - 2x_1x_6 + x_1^2) = (-3\beta+3) z^3$$

for some $z \in O_L$. We divide both sides of this equation by x_1^3 and apply Lemma 3.3 to complete the case $k = 7$.

Description of the general method. So far we have considered all the possible distributions of the prime factors $\leq k$ among the coefficients a_i. For larger values of k we use a more efficient procedure similar to that in [17]. We first outline the main ideas. We explain the important case that 3, 7, and 13 are coprime to d first.

The case $\gcd(3\cdot 7\cdot 13, d) = 1$**.** Suppose we have a solution to equation (2) with $k \geq 11$ and $\gcd(3\cdot 7, d) = 1$. Then there exist integers r_7 and r_9 such that $n \equiv r_7 d \pmod 7$ and $n \equiv r_9 d \pmod 9$. Further, we can choose the integers r_7 and r_9 to be equal; put $r := r_7 = r_9$. Then $n + id \equiv (r+i)d \pmod q$ holds for $q \in \{7, 9\}$ and $i = 0, 1 \dots, k-1$. In particular, we have $r + i \stackrel{c}{\equiv} a_i s_q \pmod q$, where $q \in \{7, 9\}$ and s_q is the inverse of d modulo q. Obviously, we may assume that $r + i$ takes values only from the set $\{-31, -30, \dots, 31\}$.

First we make a table for the residues of h modulo 7 and 9 up to cubes for $|h| < 32$, but here we present only the part with $0 \leq h < 11$.

h	0	1	2	3	4	5	6	7	8	9	10
h mod 7	0	1	2	4	4	2	1	0	1	2	4
h mod 9	0	1	2	3	4	4	3	2	1	0	1

In the first row of the table we give the values of h and in the second and third rows the corresponding residues of h modulo 7 and modulo 9 up to cubes, respectively, where the classes of the relation $\stackrel{c}{\equiv}$ are represented by $0, 1, 2, 4$ modulo 7, and by $0, 1, 2, 3, 4$ modulo 9.

Let $a_{i_1}, \dots, a_{i_t}$ be the coefficients in (3) which do not have prime divisors greater than 2. Put

$$E = \{(u_{i_j}, v_{i_j}) : r+i_j \stackrel{c}{\equiv} u_{i_j} \pmod 7, r+i_j \stackrel{c}{\equiv} v_{i_j} \pmod 9, 1 \leq j \leq t\}$$

and observe that E is contained in one of the sets

$$E_1 := \{(1,1), (2,2), (4,4)\},\ E_2 := \{(1,2), (2,4), (4,1)\},$$
$$E_3 := \{(2,1), (4,2), (1,4)\}.$$

We use this observation in the following tests which we shall illustrate by some examples.

In what follows we assume k and r to be fixed. In our method we apply the following tests in the given order. By each test some cases are eliminated.

Class cover. Let $u_i \stackrel{c}{\equiv} r + i \pmod 7$ and $v_i \stackrel{c}{\equiv} r + i \pmod 9$ $(i = 0, 1, \dots, k-1)$. For $l = 1, 2, 3$ put

$$C_l = \{i\ :\ (u_i, v_i) \in E_l,\ i = 0, 1, \dots, k-1\}.$$

Check whether the sets $C_1 \cup C_2$, $C_1 \cup C_3$, $C_2 \cup C_3$ can be covered by the multiples of the primes p with $p < k$, $p \neq 2, 3, 7$. If this is not possible for $C_{l_1} \cup C_{l_2}$, then we know that $E \subseteq E_{l_3}$ is impossible and E_{l_3} is excluded. Here $\{l_1, l_2, l_3\} = \{1, 2, 3\}$.

The forthcoming procedures are applied separately for each case where $E \subseteq E_l$ remains possible for some l. From this point on we also assume that the odd prime factors of the a_i are fixed.

Parity. Define the sets

$$I_e = \{(u_i, v_i) \in E_l \ : \ r + i \text{ is even}, \ P(a_i) \leq 2\},$$

$$I_o = \{(u_i, v_i) \in E_l \ : \ r + i \text{ is odd}, \ P(a_i) \leq 2\}.$$

As the only odd power of 2 is 1, $\min(|I_e|, |I_o|) \leq 1$ must be valid. If this does not hold, the corresponding case is excluded.

Test modulo 13. Suppose that after the previous tests we can decide whether a_i is even for the even values of i. Assume that $E \subseteq E_l$ with fixed $l \in \{1, 2, 3\}$. Further, suppose that based upon the previous tests we can decide whether a_i can be even for the even or the odd values of i. For $t = 0, 1, 2$ put

$$U_t = \{i \ : \ a_i = \pm 2^t, \ i \in \{0, 1, \ldots, k-1\}\}$$

and let

$$U_3 = \{i \ : \ a_i = \pm 5^\gamma, \ i \in \{0, 1, \ldots, k-1\}, \gamma \in \{0, 1, 2\}\}.$$

Assume that $13 \mid n + i_0 d$ for some i_0. Recall that $13 \nmid d$ and $5 \stackrel{c}{\equiv} 1 \pmod{13}$. If $i, j \in U_t$ for some $t \in \{0, 1, 2, 3\}$, then $i - i_0 \stackrel{c}{\equiv} j - i_0 \pmod{13}$. If $i \in U_{t_1}$, $j \in U_{t_2}$ with $0 \leq t_1 < t_2 \leq 2$, then $i - i_0 \stackrel{c}{\not\equiv} j - i_0 \pmod{13}$. We exclude all the cases which do not pass these tests.

Test modulo 7. Assume again that $E \subseteq E_l$ with fixed $l \in \{1, 2, 3\}$. Check whether the actual distribution of the prime divisors of the a_i yields that for some i with $7 \nmid n + id$, both $a_i = \pm t$ and $|r + i| = t$ hold for some positive integer t with $7 \nmid t$. Then

$$t \stackrel{c}{\equiv} n + id \stackrel{c}{\equiv} (r + i)d \stackrel{c}{\equiv} td \pmod{7}$$

implies that $d \stackrel{c}{\equiv} 1 \pmod{7}$. Now consider the actual distribution of the prime factors of the coefficients a_i $(i = 0, 1, \ldots, k-1)$. If in any a_i we know the exponents of all primes with one exception, and this exceptional prime p satisfies $p \stackrel{c}{\equiv} 2, 3, 4, 5 \pmod{7}$, then we can fix the exponent of p using the above information on n. As an example, assume that $7 \mid n$, and $a_1 = \pm 5^\gamma$ with $\gamma \in \{0, 1, 2\}$. Then $d \stackrel{c}{\equiv} 1 \pmod{7}$ immediately implies $\gamma = 0$. Further, if $7 \mid n$ and $a_2 = \pm 13^\gamma$

with $\gamma \in \{0,1,2\}$, then $d \overset{c}{\equiv} 1 \pmod 7$ gives a contradiction. We exclude all cases yielding a contradiction. Moreover, in the remaining cases we fix the exponents of the prime factors of the a_i-s whenever it is possible.

We remark that we used this procedure for $0 \geq r \geq -k+1$. In almost all cases it turned out that a_i is even for $r+i$ even. Further, we could prove that with $|r+i| = 1$ or 2 we have $a_i = \pm 1$ or ± 2, respectively, to conclude $d \overset{c}{\equiv} 1 \pmod 7$. The test is typically effective in case when r is "around" $-k/2$. The reason for this is that then in the sequence $r, r+1, \dots, -1, 0, 1, \dots, k-r-2, k-r-1$ several powers of 2 occur.

Induction. For fixed distribution of the prime divisors of the coefficients a_i, search for arithmetic sub-progressions of length l with $l \in \{3,5,7\}$ such that for the product Π of the terms of the sub-progression $P(\Pi) \leq L_l$ holds, with $L_3 = 2$, $L_5 = 5$, $L_7 = 7$. If there is such a sub-progression, then in view of Theorem 1.1 of [1], all such solutions can be determined.

An example. Now we illustrate how the above procedures work. For this purpose, take $k = 24$ and $r = -8$. Then, using the previous notation, we work with the following stripe (with $i \in \{0,1,\dots,23\}$):

$$\begin{array}{lcccccccccccccccccccccccc}
r+i & -8 & -7 & -6 & -5 & -4 & -3 & -2 & -1 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\
\bmod 7 & 1 & 0 & 1 & 2 & 4 & 4 & 2 & 1 & 0 & 1 & 2 & 4 & 4 & 2 & 1 & 0 & 1 & 2 & 4 & 4 & 2 & 1 & 0 & 1 \\
\bmod 9 & 1 & 2 & 3 & 4 & 4 & 3 & 2 & 1 & 0 & 1 & 2 & 3 & 4 & 4 & 3 & 2 & 1 & 0 & 1 & 2 & 3 & 4 & 4 & 3
\end{array}.$$

In the procedure *Class cover* we get the following classes:

$$C_1 = \{0,4,6,7,9,10,12,16\}, \quad C_2 = \{3,13,18\}, \quad C_3 = \{19,21\}.$$

For $p = 5, 11, 13, 17, 19, 23$ put

$$m_p = |\{i \ : \ i \in C_1 \cup C_2, \ p \mid n+id\}|,$$

respectively. Using the condition $\gcd(n,d) = 1$, one can easily check that

$$m_5 \leq 3, \ m_{11} \leq 2, \ m_{13} \leq 2, \ m_{17} \leq 1, \ m_{19} \leq 1, \ m_{23} \leq 1.$$

Hence, as $|C_1 \cup C_2| = 11$, we get that $E \subseteq E_3$ cannot be valid in this case. By a similar (but more sophisticated) calculation one gets that $E \subseteq E_2$ is also impossible. So after the procedure *Class cover* only the case $E \subseteq E_1$ remains.

From this point on, the odd prime divisors of the coefficients a_i are fixed, and we look at each case one-by-one. Observe that $p \mid n+id$ does not imply $p \mid a_i$. Further, $p \mid n+id$ implies $p \mid n+jd$ whenever $i \equiv j \pmod p$.

We consider two subcases. Suppose first that we have

$$3 \mid n+2d,\ 5 \mid n+d,\ 7 \mid n+d,\ 11 \mid n+7d,\ 13 \mid n+7d,$$

$$17 \mid n+3d,\ 19 \mid n,\ 23 \mid n+13d.$$

Then by a simple consideration we get that in *Test modulo* 13 either

$$4 \in U_1 \text{ and } 10 \in U_2,$$

or

$$10 \in U_1 \text{ and } 4 \in U_2.$$

In the first case, using $13 \mid n+7d$ we get

$$-3d \overset{c}{\equiv} 2 \pmod{13} \text{ and } 3d \overset{c}{\equiv} 4 \pmod{13},$$

which by $-3d \overset{c}{\equiv} 3d \pmod{13}$ yields a contradiction. In the second case we get a contradiction in a similar manner.

Consider now the subcase where

$$3 \mid n+2d,\ 5 \mid n+d,\ 7 \mid n+d,\ 11 \mid n+7d,\ 13 \mid n+8d,$$

$$17 \mid n+3d,\ 19 \mid n,\ 23 \mid n+13d.$$

This case survives the *Test modulo* 13. However, using the strategy explained in *Test modulo* 7, we can easily check that if a_i is even then i is even, which yields $a_9 = \pm 1$. This immediately gives $d \overset{c}{\equiv} 1 \pmod 7$. Further, we have $a_7 = \pm 11^{\varepsilon_7}$ with $\varepsilon_7 \in \{0,1,2\}$. Hence we get that

$$\pm 11^{\varepsilon_7} \overset{c}{\equiv} n+7d \overset{c}{\equiv} d \overset{c}{\equiv} 1 \pmod 7.$$

This gives $\varepsilon_7 = 0$, thus $a_7 = \pm 1$. Therefore $P(a_4a_7a_{10}) \leq 2$. Now we apply the test *Induction*.

The case $\gcd(3\cdot 7\cdot 13, d) \neq 1$. In this case we shall use the fact that almost half of the coefficients are odd. With a slight abuse of notation, when $k > 11$ we shall assume that the coefficients $a_1, a_3, \dots, a_{k-1}$ are odd, and the other coefficients are given either by $a_0, a_2, \dots, a_{k-2}$ or by $a_2, a_4, \dots, a_k$. Note that in view of $\gcd(n,d) = 1$ this can be done without loss of generality. We shall use this notation in the corresponding parts of our arguments without any further reference.

Now we continue the proof, considering the remaining cases $k \geq 11$.

The case $k = 11$. When $\gcd(3 \cdot 7, d) = 1$, the procedures *Class cover*, *Test modulo* 7 and *Induction* suffice. Hence we may suppose that $\gcd(3 \cdot 7, d) > 1$.

Assume that $7 \mid d$. Observe that $P(a_0 a_1 \dots a_4) \leq 5$ or $P(a_5 a_6 \dots a_9) \leq 5$. Hence the statement follows by induction.

Suppose next that $3 \mid d$. Observe that if $11 \nmid a_4 a_5 a_6$ then $P(a_0 a_1 \dots a_6) \leq 7$ or $P(a_4 a_5 \dots a_{10}) \leq 7$. Hence by induction and symmetry we may assume that $11 \mid a_5 a_6$. Assume first that $11 \mid a_6$. If $7 \mid a_0 a_6$ then we have $P(a_1 a_2 a_3 a_4 a_5) \leq 5$. Further, in case of $7 \mid a_5$ we have $P(a_0 a_1 a_2 a_3 a_4) \leq 5$. Thus by induction we may suppose that $7 \mid a_1 a_2 a_3 a_4$. If $7 \mid a_1 a_2 a_4$ and $5 \nmid n$, we have $P(a_0 a_5 a_{10}) \leq 2$, whence by applying Lemma 3.1 to the identity $n + (n + 10d) = 2(n + 5d)$ we get all the solutions of (2). Assume next that $7 \mid a_1 a_2 a_4$ and $5 \mid n$. Hence we deduce that one among $P(a_2 a_3 a_4) \leq 2$, $P(a_1 a_4 a_7) \leq 2$, $P(a_1 a_2 a_3) \leq 2$ is valid, and the statement follows in each case in a similar manner as above. If $7 \mid a_3$, then a simple calculation yields that one among $P(a_0 a_1 a_2) \leq 2$, $P(a_0 a_4 a_8) \leq 2$, $P(a_1 a_4 a_7) \leq 2$ is valid, and we are done. Finally, assume that $11 \mid a_5$. Then by symmetry we may suppose that $7 \mid a_0 a_1 a_4 a_5$. If $7 \mid a_4 a_5$ then $P(a_6 a_7 a_8 a_9 a_{10}) \leq 5$, and the statement follows by induction. If $7 \mid a_0$ then we have $P(a_2 a_4 a_6 a_8 a_{10}) \leq 5$, and we are done too. In case of $7 \mid a_1$ one among $P(a_0 a_2 a_4) \leq 2$, $P(a_2 a_3 a_4) \leq 2$, $P(a_0 a_3 a_6) \leq 2$ holds. This completes the case $k = 11$.

The case $k = 14$. Note that without loss of generality we may assume that $13 \mid a_i$ with $3 \leq i \leq 10$, otherwise the statement follows by induction from the case $k = 11$. Then, in particular we have $13 \nmid d$.

The tests described in the previous section suffice to dispose of the case $\gcd(3 \cdot 7 \cdot 13, d) = 1$. Assume now that $\gcd(3 \cdot 7 \cdot 13, d) > 1$ (but recall that $13 \nmid d$).

Suppose first that $7 \mid d$. Among the odd coefficients $a_1, a_3, \dots, a_{13}$ there are at most three multiples of 3, two multiples of 5 and one multiple of 11. As $13 \overset{c}{\equiv} 1 \pmod 7$, this shows that at least for one of these a_i-s we have $a_i \overset{c}{\equiv} 1 \pmod 7$. Hence $a_i \overset{c}{\equiv} 1 \pmod 7$ for every $i = 1, 3, \dots, 13$. Further, as none of $3, 5, 11$ is a cube modulo 7, we deduce that if i is odd, then either $\gcd(3 \cdot 5 \cdot 11, a_i) = 1$ or a_i must be divisible by at least two out of $3, 5, 11$. Noting that $13 \nmid d$, by Lemma 3.2 at most four numbers among $a_1, a_3, \dots, a_{13}$ can be equal to ± 1. Moreover, $\gcd(n, d) = 1$ implies that $15 \mid a_i$ can be valid for at most one $i \in \{0, 1, \dots, k-1\}$. Hence among the coefficients with odd indices there is exactly one multiple of 11, exactly one multiple of 15, and exactly one multiple of 13. Moreover, the multiple of 11 in question is also divisible either by 3 or by 5. In view of the proof of Lemma 3.2 a simple calculation yields that the cubic residues of $a_1, a_3, \dots, a_{13}$ modulo 13 must be given by $1, 1, 4, 0, 4, 1, 1$, in this order. Looking at the spots where 4 occurs in this sequence, we get that either $3 \mid a_5, a_9$ or $5 \mid a_5, a_9$ is valid. However, this contradicts the assumption $\gcd(n, d) = 1$.

Assume now that $3 \mid d$, but $7 \nmid d$. Then among the odd coefficients $a_1, a_3, \dots, a_{13}$ there are at most two multiples of 5 and one multiple of 7, 11 and 13 each. Lemma 3.2 together with $5 \overset{c}{\equiv} 1 \pmod{13}$ yields that there must be exactly four odd i-s with $a_i \overset{c}{\equiv} 1 \pmod{13}$, and further, another odd i such that a_i is divisible by 13. Hence as above, the proof of Lemma 3.2 shows that the a_i-s with odd indices are $\overset{c}{\equiv} 1, 1, 4, 0, 4, 1, 1 \pmod{13}$, in this order. As the prime 11 should divide an a_i with odd i and $a_i \overset{c}{\equiv} 4 \pmod{13}$, this yields that $11 \mid a_5 a_9$. However, as above, this immediately yields that $P(a_0 a_2 \dots a_{12}) \le 7$ (or $P(a_2 a_4 \dots a_{14}) \le 7$), and the case $k = 14$ follows by induction.

The case $k = 18$. Using the procedures described in the previous section, the case $\gcd(3 \cdot 7 \cdot 13, d) = 1$ can be excluded. So we may assume $\gcd(3 \cdot 7 \cdot 13, d) > 1$.

Suppose first that $7 \mid d$. Among $a_1, a_3, \dots, a_{17}$ there are at most three multiples of 3, two multiples of 5 and one multiple of 11, 13 and 17 each. Hence at least for one odd i we have $a_i = \pm 1$. Thus all of $a_1, a_3, \dots, a_{17}$ are $\overset{c}{\equiv} 1 \pmod{7}$. Among the primes $3, 5, 11, 13, 17$ only 13 is $\overset{c}{\equiv} 1 \pmod{7}$, so the other primes cannot occur alone. Hence we get that $a_i = \pm 1$ for at least five out of $a_1, a_3, \dots, a_{17}$. However, by Lemma 3.2 this is possible only if $13 \mid d$. In that case $a_i = \pm 1$ holds for at least six coefficients with i odd. Now a simple calculation shows that among them three are in arithmetic progression. This leads to an equation of the shape $X^3 + Y^3 = 2Z^3$, and Lemma 3.1 applies.

Assume next that $13 \mid d$, but $7 \nmid d$. Among the odd coefficients $a_1, a_3, \dots a_{17}$ there are at most three multiples of 3, two multiples of 5 and 7 each, and one multiple of 11 and 17 each. Hence, by $5 \overset{c}{\equiv} 1 \pmod{13}$ there are at least two $a_i \overset{c}{\equiv} 1 \pmod{13}$, whence all $a_i \overset{c}{\equiv} 1 \pmod{13}$. As from this list only the prime 5 is a cube modulo 13, we get that at least four out of the above nine odd a_i-s are equal to ± 1. Recall that $7 \nmid d$ and observe that the cubic residues modulo 7 of a seven-term arithmetic progression with common difference not divisible by 7 is a cyclic permutation of one of the sequences

$$0, 1, 2, 4, 4, 2, 1, \quad 0, 2, 4, 1, 1, 4, 2, \quad 0, 4, 1, 2, 2, 1, 4.$$

Hence remembering that for four odd i we have $a_i = \pm 1$, we get that the cubic residues of $a_1, a_3, \dots, a_{17}$ modulo 7 are given by $1, 1, 4, 2, 0, 2, 4, 1, 1$, in this order. In particular, we have exactly one multiple of 7 among them. Further, looking at the spots where $0, 2$ and 4 occur, we deduce that at most two of the a_i-s with odd indices can be multiples of 3. Switching back to modulo 13, this yields that $a_i = \pm 1$ for at least five a_i-s. However, this contradicts Lemma 3.2.

Finally, assume that $3 \mid d$. In view of what we have proved already, we may further suppose that $\gcd(7 \cdot 13, d) = 1$. Among the odd coefficients $a_1, a_3, \dots, a_{17}$ there are at most two multiples of 5 and 7 each, and one multiple of 11, 13 and 17 each. Hence as $7 \nmid d$ and $13 \overset{c}{\equiv} 1 \pmod 7$, we get that the cubic residues modulo 7 of the coefficients a_i with odd i are given by one of the sequences

$$1, 0, 1, 2, 4, 4, 2, 1, 0, \quad 0, 1, 2, 4, 4, 2, 1, 0, 1, \quad 1, 1, 2, 4, 0, 4, 2, 1, 1.$$

In view of the places of the values 2 and 4, we see that it is not possible to distribute the prime divisors $5, 7, 11$ over the a_i-s with odd indices. This finishes the case $k = 18$.

The case $k = 20$. By the help of the procedures described in the previous section, in case of $\gcd(3 \cdot 7 \cdot 13, d) = 1$ all solutions to equation (2) can be determined. Assume now that $\gcd(3 \cdot 7 \cdot 13, d) > 1$.

We start with the case $7 \mid d$. Then among the odd coefficients $a_1, a_3, \dots, a_{19}$ there are at most four multiples of 3, two multiples of 5, and one multiple of 11, 13, 17 and 19 each. As $13 \overset{c}{\equiv} 1 \pmod 7$, this yields that $a_i \overset{c}{\equiv} 1 \pmod 7$ for all i. Hence the primes $3, 5, 11, 17, 19$ must occur at least in pairs in the a_i-s with odd indices, which yields that at least five such coefficients are equal to ± 1. Thus Lemma 3.2 gives $13 \mid d$, whence $a_i \overset{c}{\equiv} 1 \pmod{13}$ for all i. Hence we deduce that the prime 5 may be only a third prime divisor of the a_i-s with odd indices, and so at least seven out of $a_1, a_3, \dots, a_{19}$ equal ± 1. However, then there are three such coefficients which belong to an arithmetic progression. Thus by Lemma 3.1 we get all solutions in this case.

Assume next that $13 \mid d$. Without loss of generality we may further suppose that $7 \nmid d$. Then among the odd coefficients $a_1, a_3, \dots, a_{19}$ there are at most four multiples of 3, two multiples of 5 and 7 each, and one multiple of 11, 17 and 19 each. As $5 \overset{c}{\equiv} 1 \pmod{13}$ this implies $a_i \overset{c}{\equiv} 1 \pmod{13}$ for all i, whence the primes $3, 7, 11, 17, 19$ should occur at least in pairs in the a_i-s with odd i. Hence at least four of these coefficients are equal to ± 1. By a similar argument as in case of $k = 18$, we get that the cubic residues of $a_1, a_3, \dots, a_{19}$ modulo 7 are given by one of the sequences

$$1,0,1,2,4,4,2,1,0,1, \quad 1,1,4,2,0,2,4,1,1,4, \quad 4,1,1,4,2,0,2,4,1,1.$$

In view of the positions of the $0,2$ and 4 values, we get that at most two corresponding terms can be divisible by 3 in the first case, which modulo 13 yields that the number of odd i-s with $a_i = \pm 1$ is at least five. This is a contradiction modulo 7. Further, in the last two cases at most three terms can be divisible by 3, and exactly one term is a multiple of 7. This yields modulo 13 that the number of odd i-s with $a_i = \pm 1$ is at least five, which is a contradiction modulo 7 again.

Finally, suppose that $3 \mid d$. We may assume that $\gcd(7 \cdot 13, d) = 1$. Then among the odd coefficients $a_1, a_3, \ldots, a_{19}$ there are at most two multiples of 5 and 7 each, and one multiple of 11, 13, 17 and 19 each. Hence Lemma 3.2 yields that exactly four of these coefficients should be $\stackrel{c}{\equiv} 1 \pmod{13}$, and exactly one of them must be a multiple of 13. Further, exactly two other a_i-s with odd indices are multiples of 7, and these a_i-s are divisible by none of $11, 13, 17, 19$. So in view of the proof of Lemma 3.2 a simple calculation gives that the cubic residues of $a_1, a_3, \ldots, a_{19}$ modulo 13 are given by one of the sequences

$$0,2,4,4,1,2,1,1,2,1, \quad 1,2,1,1,2,1,4,4,2,0,$$

$$2,4,2,1,1,4,0,4,1,1, \quad 1,1,4,0,4,1,1,2,4,2.$$

In the upper cases we get that 7 divides two terms with $a_i \stackrel{c}{\equiv} 2 \pmod{13}$, whence the power of 7 should be 2 in both cases. However, this implies $7^2 \mid 14d$, hence $7 \mid d$, a contradiction. As the lower cases are symmetric, we may assume that the very last possibility occurs. In that case we have $7 \mid a_5$ and $7 \mid a_{19}$. We may assume that $11 \mid a_{17}$, otherwise $P(a_6 a_8 \ldots a_{18}) \leq 7$ and the statement follows by induction. Further, we also have $13 \mid a_7$, and $17 \mid a_9$ and $19 \mid a_{15}$ or vice versa. Hence either $P(a_3 a_8 a_{13}) \leq 2$ or $P(a_4 a_{10} a_{16}) \leq 2$, and induction suffices to complete the case $k = 20$.

The case $k = 24$. The procedures described in the previous section suffice to completely treat the case $\gcd(3 \cdot 7 \cdot 13, d) = 1$. So we may assume that $\gcd(3 \cdot 7 \cdot 13, d) > 1$ is valid.

Suppose first that $7 \mid d$. Among the odd coefficients $a_1, a_3, \ldots, a_{23}$ there are at most four multiples of 3, three multiples of 5, two multiples of 11, and one multiple of 13, 17, 19 and 23 each. We know that all a_i belong to the same cubic class modulo 7. As $3 \stackrel{c}{\equiv} 4 \pmod 7$, $5 \stackrel{c}{\equiv} 2 \pmod 7$ and among the coefficients $a_1, a_3, \ldots, a_{23}$ there are at most two multiples of 3^2 and at most one multiple of 5^2, we get that these coefficients are all $\stackrel{c}{\equiv} 1 \pmod 7$. This yields that the primes

$3, 5, 11, 17, 19, 23$ may occur only at least in pairs in the coefficients with odd indices. Thus we get that at least five out of $a_1, a_3, \dots, a_{23}$ are $\stackrel{c}{\equiv} 1 \pmod{13}$. Hence, by Lemma 3.2 we get that $13 \mid d$ and consequently $a_i \stackrel{c}{\equiv} 1 \pmod{13}$ for all i. This also shows that the 5-s can be at most third prime divisors of the a_i-s with odd indices. So we deduce that at least eight out of the odd coefficients $a_1, a_3, \dots, a_{23}$ are equal to ± 1. However, a simple calculation shows that from the eight corresponding terms we can always choose three forming an arithmetic progression. Hence this case follows from Lemma 3.1.

Assume next that $13 \mid d$, but $7 \nmid d$. Among the coefficients with odd indices there are at most four multiples of 3, three multiples of 5, two multiples of 7 and 11 each, and one multiple of 17, 19 and 23 each. Hence, by $5 \stackrel{c}{\equiv} 1 \pmod{13}$ we deduce $a_i \stackrel{c}{\equiv} 1 \pmod{13}$ for all i. As before, a simple calculation yields that at least for four of these odd coefficients $a_i = \pm 1$ hold. Hence looking at the possible cases modulo 7, one can easily see that we cannot have four multiples of 3 at the places where $0, 2$ and 4 occur as cubic residues modulo 7. Hence in view of Lemma 3.2 we need to use two 11-s, which yields that $11 \mid a_1$ and $11 \mid a_{23}$. Thus the only possibility for the cubic residues of $a_1, a_3, \dots, a_{23}$ modulo 7 is given by the sequence

$$2, 1, 0, 1, 2, 4, 4, 2, 1, 0, 1, 2.$$

However, the positions of the 2-s and 4-s allow to have at most two a_i-s with odd indices which are divisible by 3 but not by 7. Hence switching back to modulo 13, we get that there are at least five a_i-s which are ± 1, a contradiction by Lemma 3.2.

Finally, assume that $3 \mid d$, and $\gcd(7 \cdot 13, d) = 1$. Then among $a_1, a_3, \dots, a_{23}$ there are at most three multiples of 5, two multiples of 7 and 11 each, and one multiple of 13, 17, 19 and 23 each. Hence by Lemma 3.2 we get that exactly four of the coefficients $a_1, a_3, \dots, a_{23}$ are $\stackrel{c}{\equiv} 1 \pmod{13}$, and another is a multiple of 13. Further, all the mentioned prime factors (except the 5-s) divide distinct a_i-s with odd indices. Using that at most these coefficients can be divisible by 7^2 and 11^2, in view of the proof of Lemma 3.2 we get that the only possibilities for the cubic residues of these coefficients modulo 13 are given by one of the sequences

$$2, 2, 4, 2, 1, 1, 4, 0, 4, 1, 1, 2, \quad 2, 1, 1, 4, 0, 4, 1, 1, 2, 4, 2, 2.$$

By symmetry we may assume the first possibility. Then we have $7 \mid a_3$, $11 \mid a_1$, $13 \mid a_{15}$, and $17, 19, 23$ divide a_5, a_7, a_{13} in some order. Hence

$P(a_4a_9a_{14}) \le 2$, or $5 \mid n+4d$ whence $P(a_{16}a_{18}a_{20}) \le 2$. In both cases we apply induction.

The case $k = 30$. By the help of the procedures described in the previous section, the case $\gcd(3\cdot7\cdot13, d) = 1$ can be excluded. Assume now that $\gcd(3\cdot7\cdot13, d) > 1$.

We start with the case $7 \mid d$. Then among the odd coefficients $a_1, a_3, \dots, a_{29}$ there are at most five multiples of 3, three multiples of 5, two multiples of 11 and 13 each, and one multiple of 17, 19, 23 and 29 each. As $13 \overset{c}{\equiv} 29 \overset{c}{\equiv} 1 \pmod 7$, this yields that $a_i \overset{c}{\equiv} 1 \pmod 7$ for all i. Hence the other primes must occur at least in pairs in the a_i-s with odd indices, which yields that at least six such coefficients are equal to ±1. Further, we get that the number of such coefficients $\overset{c}{\equiv} 0, 1 \pmod{13}$ is at least eight. However, by Lemma 3.2 this is possible only if $13 \mid d$, whence $a_i \overset{c}{\equiv} 1 \pmod{13}$ for all i. Then 5 and 29 can be at most third prime divisors of the coefficients a_i-s with odd i-s. So a simple calculation gives that at least ten out of the odd coefficients $a_1, a_3, \dots, a_{29}$ are equal to ±1. Hence there are three such coefficients in arithmetic progression, and the statement follows from Lemma 3.1.

Assume next that $13 \mid d$, but $7 \nmid d$. Then among the odd coefficients $a_1, a_3, \dots, a_{29}$ there are at most five multiples of 3, three multiples of 5 and 7 each, two multiples of 11, and one multiple of 17, 19, 23 and 29 each. From this we get that $a_i \overset{c}{\equiv} 1 \pmod{13}$ for all i. Hence the primes different from 5 should occur at least in pairs. We get that at least five out of the coefficients $a_1, a_3, \dots, a_{29}$ are equal to ±1. Thus modulo 7 we get that it is impossible to have three terms divisible by 7. Then it follows modulo 13 that at least six a_i-s with odd indices are equal to ±1. However, this is possible only if $7 \mid d$, which is a contradiction.

Finally, assume that $3 \mid d$, but $\gcd(7\cdot13, d) = 1$. Then among the odd coefficients $a_1, a_3, \dots, a_{29}$ there are at most three multiples of 5 and 7 each, two multiples of 11 and 13 each, and one multiple of 17, 19, 23 and 29 each. Further, modulo 7 we get that all primes $5, 11, 17, 19, 23$ divide distinct a_i-s with odd indices, and the number of odd i-s with $a_i \overset{c}{\equiv} 0, 1 \pmod 7$ is seven. However, checking all possibilities modulo 7, we get a contradiction. This completes the proof of Theorem 2.2. □

Proof of Theorem 2.1. Obviously, for $k < 32$ the statement is an immediate consequence of Theorem 2.2. Further, observe that $b = 1$ implies that for any k with $31 < k < 39$, one can find j with $0 \le j \le k - 30$ such that $P(a_ja_{j+1}\dots a_{j+29}) \le 29$. Hence the statement follows from Theorem 2.2. □

References

[1] M. Bennett, N. Bruin, K. Győry, L. Hajdu, *Powers from products of consecutive terms in arithmetic progression*, Proc. London Math. Soc. **92** (2006), 273–306.

[2] W. Bosma, J. Cannon and C. Playoust, *The Magma algebra system. I. The user language*, J. Symbolic Comput. **24** (1997), 235–265.

[3] B. Brindza, L. Hajdu, I. Z. Ruzsa, *On the equation* $x(x+d)\dots(x+(k-1)d) = by^2$, Glasgow Math. J. 42 (2000), 255–261.

[4] N. Bruin, *Chabauty methods and covering techniques applied to generalized Fermat equations*, CWI Tract, Vol. 133, Stichting Mathematisch Centrum, Centrum voor Wiskunde en Informatica, Amsterdam, 2002.

[5] N. Bruin, *Chabauty methods using elliptic curves*, J. Reine Angew. Math. **562** (2003), 27–49.

[6] H. Darmon, L. Merel, *Winding quotients and some variants of Fermat's Last Theorem*, J. Reine Angew. Math. **490** (1997), 81–100.

[7] L. E. Dickson, *History of the theory of numbers. Vol. II: Diophantine analysis*, Chelsea Publishing Co., New York 1966, xxv+803 pp.

[8] P. Erdős, *Note on products of consecutive integers (II)*, J. London Math. Soc. **14** (1939), 194–198.

[9] P. Erdős, J. L. Selfridge, *The product of consecutive integers is never a power*, Illinois J. Math. **19** (1975), 292–301.

[10] P. Filakovszky, L. Hajdu, *The resolution of the equation* $x(x+d)\dots(x+(k-1)d) = by^2$ *for fixed* d, Acta Arith. **98** (2001), 151–154.

[11] K. Győry, *On the diophantine equation* $\binom{n}{k} = x^l$, Acta Arith. **80** (1997), 289–295.

[12] K. Győry, *On the diophantine equation* $n(n+1)\dots(n+k-1) = bx^l$, Acta Arith. **83** (1998), 87–92.

[13] K. Győry, *Power values of products of consecutive integers and binomial coefficients*, Number Theory and Its Applications (S. Kanemitsu and K. Győry, eds.), Kluwer Acad. Publ. 1999, pp. 145–156.

[14] K. Győry, *Perfect powers in products with consecutive terms from arithmetic progressions*, More Sets, Graphs and Numbers (E. Győri, G. O. H Katona, L. Lovász, eds.), Bolyai Society Mathematical Studies **15** (2006), pp. 143-155.

[15] K. Győry, L. Hajdu, N. Saradha, *On the Diophantine equation* $n(n+d)\dots(n+(k-1)d) = by^l$, Canad. Math. Bull. **47** (2004), 373–388; **48** (2005), 636.

[16] G. Hanrot, N. Saradha, T. N. Shorey, *Almost perfect powers in consecutive integers*, Acta Arith. **99** (2001), 13–25.

[17] N. Hirata-Kohno, S. Laishram, T. N. Shorey, R. Tijdeman, *An extension of a theorem of Euler*, Acta Arith. **129** (2007), 71–102.

[18] S. Laishram, *An estimate for the length of an arithmetic progression the product of whose terms is almost square*, Publ. Math. Debrecen **68** (2006), 451–475.

[19] S. Laishram, T. N. Shorey, *The equation* $n(n+d)\dots(n+(k-1)d) = by^2$ *with* $\omega(d) \le 6$ *or* $d \le 10^{10}$, Acta Arith. **129** (2007), 249–305.

[20] R. Marszalek, *On the product of consecutive terms of an arithmetic progression*, Monatsh. Math. **100** (1985), 215–222.

[21] R. Obláth, *Über das Produkt fünf aufeinander folgender Zahlen in einer arithmetischer Reihe*, Publ. Math. Debrecen **1** (1950), 222–226.

[22] O. Rigge, *Über ein diophantisches Problem*, 9th Congress Math. Scand., Helsingfors 1938, Mercator 1939, pp. 155–160.

[23] J. W. Sander, *Rational points on a class of superelliptic curves*, J. London Math. Soc. **59** (1999), 422–434.

[24] N. Saradha, *On perfect powers in products with terms from arithmetic progressions*, Acta Arith. **82** (1997), 147–172.

[25] N. Saradha, *Squares in products with terms in an arithmetic progression*, Acta Arith. **86** (1998), 27–43.

[26] N. Saradha, T. N. Shorey, *Almost perfect powers in arithmetic progression*, Acta Arith. **99** (2001), 363–388.

[27] N. Saradha, T. N. Shorey, *Almost Squares in Arithmetic Progression*, Compositio Math. **138** (2003), 73–111.

[28] N. Saradha, T. N. Shorey, *Almost Squares and Factorisations in Consecutive Integers*, Compositio Math. **138** (2003), 113-124.

[29] N. Saradha, T. N. Shorey, *Contributions towards a conjecture of Erdős on perfect powers in arithmetic progressions*, Compositio Math. **141** (2005), 541-560.

[30] E. Selmer, *The diophantine equation $ax^3 + by^3 + cz^3 = 0$*, Acta Math. **85** (1951), 205–362.

[31] T. N. Shorey, *Perfect powers in products of arithmetical progressions with fixed initial term*, Indag. Math. N.S. **7** (1996), 521–525.

[32] T. N. Shorey, *Powers in arithmetic progression*, A Panorama in Number Theory (G. Wüstholz, ed.), Cambridge University Press 2002, pp. 325–336.

[33] T. N. Shorey, *Powers in arithmetic progression (II)*, New Aspects of Analytic Number Theory, Kyoto 2002, pp. 202–214.

[34] T. N. Shorey, *Diophantine approximations, Diophantine equations, Transcendence and Applications*, Indian Jour. of Pure and Applied Math. **37** (2006), 9-39.

[35] T. N. Shorey, R. Tijdeman, *Perfect powers in products of terms in an arithmetic progression*, Compositio Math. **75** (1990), 307–344.

[36] T. N. Shorey, R. Tijdeman, *Perfect powers in products of terms in an arithmetic progression II*, Compositio Math. **82** (1992), 119–136.

[37] T. N. Shorey, R. Tijdeman, *Perfect powers in products of terms in an arithmetic progression III*, Acta Arith. **61** (1992), 391–398.

[38] T. N. Shorey, R. Tijdeman, *Some methods of Erdős applied to finite arithmetic progressions integers and related equations*, The Mathematics of Paul Erdős I, Springer 1997, pp. 251–267.

[39] Sz. Tengely, *Note on the paper "An extension of a theorem of Euler" by Hirata-Kohno et. al*, Acta Arith. **134** (2008), 329–335.

[40] R. Tijdeman, *Diophantine equations and diophantine approximations*, Number Theory and Applications, Kluwer Acad. Press 1989, 215–243.

[41] R. Tijdeman, *Exponential diophantine equations 1986-1996*, Number Theory: Diophantine, Computational and Algebraic Aspects (K. Győry, A. Pethő and V. T. Sós, eds.), Walter de Gruyter, Berlin–New York 1998, pp. 523–540.

Lajos Hajdu
Institute of Mathematics
University of Debrecen
and the Number Theory Research Group
of the Hungarian Academy of Sciences
P.O.Box 12
4010 Debrecen
Hungary
E-mail address: hajdul@math.klte.hu

Szabolcs Tengely
Institute of Mathematics
University of Debrecen
and the Number Theory Research Group
of the Hungarian Academy of Sciences
P.O.Box 12
4010 Debrecen
Hungary
E-mail address: tengely@math.klte.hu

Robert Tijdeman
Mathematical Institute
Leiden University
P.O.Box 9512
2300 RA Leiden
The Netherlands
E-mail address: tijdeman@math.leidenuniv.nl