ON THE DIOPHANTINE EQUATION $F_n = P(x)$

SZ. TENGELY AND M. ULAS

Dedicated to Professor Attila Pethő on his 70th birthday.

Abstract. We consider equations of the form $F_n = P(x)$, where P is a polynomial with integral coefficients and F_n is the nth Fibonacci number that is, $F_0 = 0, F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n > 1$. In particular, for each $k \in \mathbb{N}_+$, we prove the existence of a polynomial $F_k \in \mathbb{Z}[x]$ of degree $2k-1$ such that the Diophantine equation $F_k(x) = F_m$ has infinitely many solutions in positive integers (x, m). Moreover, we present results of our numerical search concerning the existence of even degree polynomials representing many Fibonacci numbers. We also determine all integral solutions (n, x) of the Diophantine equations $\binom{x}{2} + d = F_n \quad$ for $-20 \le d \le 20$ and $F_n = \binom{x}{5}$.

1. Introduction

Let R_n be a linear recursive sequence and $P \in \mathbb{Z}[X]$ be a polynomial. We consider Diophantine equations of the form $P(x) = R_n$. In the literature there have been many nice articles published identifying perfect powers, products of consecutive integers, binomial coefficients, figurate numbers and power sums in linear recursive sequences like the Fibonacci sequence, Lucas sequence and Pell sequence defined as follows

$$\begin{aligned} F_0 &= 0, \quad F_1 = 1, \quad F_n = F_{n-1} + F_{n-2} \text{ for } n \ge 2, \\ L_0 &= 2, \quad L_1 = 1, \quad L_n = L_{n-1} + L_{n-2} \text{ for } n \ge 2, \\ P_0 &= 0, \quad P_1 = 1, \quad P_n = 2P_{n-1} + P_{n-2} \text{ for } n \ge 2. \end{aligned}$$

A result obtained by Ljunggren [17] implies that the only squares in the Fibonacci sequence are $F_0 = 0, F_1 = F_2 = 1, F_{12} = 144$. Cohn [9, 10] and Wyler [31] rediscovered this statement. In case of the Lucas sequence the perfect squares were determined by Alfred [1] and Cohn [11]. Pethő [22] and independently Cohn [12] provided the complete list of perfect squares in the Pell sequence. Cubes and higher powers were considered by London and Finkelstein [18] and Pethő [20],[21]. Bugeaud, Mignotte and Siksek [8] combined the Baker's method, modular approach and some classical techniques to show that the perfect powers in the Fibonacci sequence are 0,1,8 and 144, and the perfect powers in the Lucas sequence are 1 and 4. In cases of the Diophantine equations

$$F_n = \binom{x}{k}, \qquad L_n = \binom{x}{k}, \qquad P_n = \binom{x}{k}$$

Szalay [27],[28] obtained results if $k = 3$ and Kovács [16] if $k = 4$. Tengely [29] determined the solutions in case of L_n with $k = 5$. Diophantine equations of the

2000 *Mathematics Subject Classification.* Primary 11D61; Secondary 11Y50.
Key words and phrases. Diophantine equations, recurrence sequences.

forms

$$F_n = x^p \pm 1 \qquad \text{and} \qquad F_n = x^p \pm 2,$$

were solved in [3] and [6]. Let A, B, R_0, R_1 be integers. A binary linear recurrence sequence R_n is defined by two initial values (R_0, R_1) and by the relation

$$R_{n+1} = AR_n - BR_{n-1}, n \geq 1.$$

Such a sequence is called non-degenerate if $|R_0| + |R_1| > 0$ and the quotient of the roots, $\alpha_1, \alpha_2 \in \mathbb{C}$ of the characteristic polynomial of R_n (defined by $x^2 - Ax + B$) is not a root of unity. Let us introduce some additional notation. Let $D = A^2 - 4B$ and $C = R_1^2 - AR_1R_0 + BR_0^2$. Let $T_k(x)$ denote the Chebishev polynomial of degree k, defined by $T_0(x) = 2, T_1(x) = x$ and $T_{n+1}(x) = xT_n(x) - T_{n-1}(x)$ for $n \geq 1$. The following elegant characterization is due to Nemes and Pethő [19].

Theorem (Nemes-Pethő)**.** *Let R_n be a non-degenerated second order recurrence with $|B| = 1$, and $P = \sum_{k=0}^{d} A_k X^k$ be a polynomial with integer coefficients of degree $d \geq 2$. Let be $q = -B^m C/D$ and $E = 2(d-1)A_{d-1}^2 - 4dA_dA_{d-2}$. If the equation $R_n = P(x)$ has infinitely many integer solutions n, x, then*

$$P(x) = \varepsilon\sqrt{q}T_d\left(\frac{2d|A_d|}{\eta\sqrt{E}} + \frac{2A_{d-1}}{\eta\sqrt{E}}\right),$$

where ε and η are either 1 or -1. Furthermore, either x is an integer root of $P'(x)$ or $d|A_d|x + A_{d-1}$ is contained in the union of finitely many second order recurrence sequences with discriminants D_i, where D/D_i are squares of integers.

Based on this result Nemes and Pethő noted that the equation $F_n = P(x)$ can have infinitely many solutions only when the degree of P is odd. They also remarked that the assumptions of the theorem are not sufficient, i.e., the equation $R_n = P(x)$ may have finitely many solutions even if the polynomial is of the expected form. In fact they noted that the equation $L_n = P(x) = 3x^2 - 2 = T_2(\sqrt{3}x)$ although of expected form, has no integer solutions. Thus, it is quite natural to ask whether we can construct an explicit form of polynomial P such that the equation $L_n = P(x)$ has infinitely many integer solutions. Similar question can be asked for the sequence of Fibonacci numbers $(F_n)_{n\in\mathbb{N}}$. As we will see such construction can be performed for quite general class of Lucas sequences, with particular examples being Fibonacci and Lucas sequences. Indeed, in Section 2 we consider the sequences $(P_n(a))_{n\in\mathbb{N}}, (Q_n(a))_{n\in\mathbb{N}}$, where $a \in \mathbb{C} \setminus \{-1, 0, 1\}$ and

$$P_n(a) = \frac{a^n - a^{-n}}{a - a^{-1}}, \quad Q_n(a) = a^n + a^{-n}.$$

We show that for each $k \in \mathbb{N}_+$ there is a square-free polynomial $F_k(a,t) \in \mathbb{Q}(a)[t]$ of degree $2k-1$ such that the Diophantine equation $F_k(a,t) = P_m(a)$ has infinitely many solutions in positive integers t, m. Similarly, we prove that for each $k \in \mathbb{N}_+$ there is a polynomial $G_k \in \mathbb{Z}[t]$ of degree $k+1$ such that the Diophantine equation $G_k(t) = Q_m(a)$ has infinitely many solutions in positive integers t, m. By an appropriate specialization $a = a_0$ we get polynomials $F_k(a_0, t), G_k(t) \in \mathbb{Z}[t]$ such that the equations

$$F_k(a_0, t) = F_n, \quad G_k(t) = L_n$$

have infinitely many solutions in integers. Moreover, an additional advantage of our construction is the possibility to determine the explicit form of the discriminants of our polynomials. In Section 3 we present results of our numerical calculations concerning the existence of degree two polynomials f such that the Diophantine equation $f(x) = F_m$, where F_m is the mth Fibonacci number, has at least four solutions in integers x, m. Moreover, based on our computations we state several conjectures and certain general problems

Finally, in the last section we characterize integral solutions of several Diophantine equations related to representations of Fibonacci numbers by shifted triangular numbers, that is we resolve the equations

$$\binom{x}{2} + d = F_n \quad \text{for } -20 \leq d \leq 20.$$

Moreover, by investigating certain genus two curves we characterize all integral solutions of the Diophantine equation $\binom{x}{5} = F_n$. Our result complement earlier findings concerning integer solutions of the equation $\binom{x}{k} = F_n$, where $k \leq 4$.

2. Polynomials representing infinitely many Fibonacci and related numbers

Which polynomials $P \in \mathbb{Z}[X]$ represent many different Fibonacci numbers? For a given polynomial we may expect only finitely many solutions of the equation $P(x) = F_n$. Indeed, we recall the identity $L_n^2 = 5F_n^2 \pm 4$ and thus, if we are interested in finding integer solutions of the equation $P(x) = F_n$, then it is enough to find all integer solutions of the Diophantine equation

$$y^2 = 5P(x)^2 \pm 4.$$

The above equation defines a hyperelliptic curve, say C. From Siegel theorem we know that if the polynomial $5P(x)^2 \pm 4$ can not be represented in the form $f_1(x)^2 f_2(x)$ for certain polynomials $f_1, f_2 \in \mathbb{Q}[x]$, where f_2 is a square-free polynomial of degree ≥ 3, then there are only finitely many integral points on the curve C. In consequence, we have only finitely many solutions of the related equation $P(x) = F_n$.

We would like to determine polynomials P_d of a given degree $d > 1$ such that the set $\{P_d(x) : x \in \mathbb{Z}\}$ contains many Fibonacci numbers.

In case of $d = 2$ the polynomials $P_{2,k}(x) = 3x^2 + (6k+2)x + k(3k+2)$ represent the Fibonacci numbers $0, 1, 5, 8, 21, 4181$ since

$$\begin{aligned} &P_{2,k}(-k) = 0, P_{2,k}(-k-1) = 1, P_{2,k}(-k+1) = 5, \\ &P_{2,k}(-k-2) = 8, P_{2,k}(-k-3) = 21, P_{2,k}(-k+37) = 4181. \end{aligned}$$

Here we remark that obviously the above family comes from a given polynomial, namely $3x^2 + 2x$ by applying the substitution $x := x + k$. Therefore, if we consider the question of representability of a given number by a polynomial $P_d(x) = A_d x^d + A_{d-1}x^{d-1} + \ldots$ of degree d with $A_d > 0$, then without loss of generality we can assume that $|A_{d-1}| \leq dA_d$. Indeed, we can always write $A_{d-1} = dA_d k + r$ for some $k \in \mathbb{Z}$ with $0 \leq r < dA_d$ and thus after the substitution $x := x + k$ we get a polynomial in the required form.

It is clear from Lagrange interpolation formula that for a given d we may construct a polynomial of degree d representing $d+1$ Fibonacci numbers. For example in case of $d = 3$ we use the points $(0,0), (1,1), (2,2), (3, F_n)$ to obtain the polynomial

$$\left(\frac{1}{6}F_n - \frac{1}{2}\right)x^3 + \left(-\frac{1}{2}F_n + \frac{3}{2}\right)x^2 + \frac{1}{3}F_n x.$$

The polynomial has integral coefficients if $n \equiv 0 \pmod 4$ and $n \not\equiv 0 \pmod 3$. Hence we may take $F_{16} = 987$ to get $164x^3 - 492x^2 + 329x$. The latter polynomial represents the Fibonacci numbers $0, 1, 2, 987$.

We prove that there exist odd degree polynomials representing infinitely many Fibonacci numbers and related sequences.

Theorem 2.1. *For $a \in \mathbb{C} \setminus \{-1, 0, 1\}$ let us consider the sequence $(P_n(a))_{n\in\mathbb{N}}$, where*

$$P_n = P_n(a) = \frac{a^n - a^{-n}}{a - a^{-1}}.$$

Then, for any given $k \in \mathbb{N}_+$ there is a square-free polynomial $F_k(a,t) \in \mathbb{Z}\left[\frac{1}{2}\right][t]$ of degree $2k-1$ such that the Diophantine equation $F_k(a,t) = P_n$ has infinitely many solutions in integers t, n.

The most difficult part of the proof of the above theorem is square-freeness of polynomial $F_k(a,t)$. In order to do that we will compute discriminant of the polynomial $F_k(a,t)$. Thus, we recall below the notion of a resultant of two polynomials and a discriminant.

Let K be a field and consider the polynomials $F, G \in K[x]$ given by

$$\begin{aligned} F(x) &= a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0, \\ G(x) &= b_m x^m + b_{m-1}x^{m-1} + \ldots + b_1 x + b_0. \end{aligned} \tag{1}$$

The resultant of the polynomials F, G is defined as

$$\mathrm{Res}(F,G) = a_n^m b_m^n \prod_{i=1}^{n}\prod_{j=1}^{m}(\alpha_i - \beta_j),$$

where $\alpha_1, \ldots, \alpha_n$ and $\beta_1, \ldots, \beta_m$ are the roots of F and G respectively (viewed in an appropriate field extension of K). We define the discriminant of the polynomial F in the following way:

$$\mathrm{Disc}(F) = \frac{(-1)^{\frac{n(n-1)}{2}}}{a_n}\mathrm{Res}(F, F').$$

We collect basic properties of the resultant of the polynomials F, G:

$$\mathrm{Res}(F,G) = a_n^m \prod_{i=1}^{n} G(\alpha_i) = b_m^n \prod_{i=1}^{m} F(\beta_i), \tag{2}$$

$$\mathrm{Res}(F,G) = (-1)^{nm}\,\mathrm{Res}(G,F), \tag{3}$$

$$\mathrm{Res}(F, G_1G_2) = \mathrm{Res}(F,G_1)\,\mathrm{Res}(F,G_2). \tag{4}$$

Moreover, if $F(x) = a_0$ is a constant polynomial then, unless $F = G = 0$, we have

$$\mathrm{Res}(F,G) = \mathrm{Res}(a_0, G) = \mathrm{Res}(G, a_0) = a_0^m. \tag{5}$$

Finally, we recall an important result concerning the formula for the resultant of the polynomial G and F, provided that $F(x) = q(x)G(x) + r(x)$. More precisely, we have the following.

Lemma 2.2. *Let $F, G \in K[x]$ be given by* (1) *and suppose that $F(x) = q(x)G(x) + r(x)$ for some $q, r \in K[x]$. Then we have the formula*

$$\mathrm{Res}(G,F) = b_m^{\deg F - \deg r}\,\mathrm{Res}(G, r).$$

The proof of the above lemma can be found in [23] (see also [13]).

Proof of Theorem 2.1. We define the sequence $(F_k(a,t))_{n\in\mathbb{N}_+}$ of polynomials in a recursive way. More precisely, we put

$$F_1(a,t)=t,\ F_2(a,t)=\frac{1}{a^2}t((a^2-1)^2t^2+3a^2),$$
$$F_k(a,t)=\frac{(a^2-1)^2t^2+2a^2}{a^2}F_{k-1}(a,t)-F_{k-2}(a,t),\ k\geq 3,$$

and prove that for each $k,n\in\mathbb{N}_+$ the following identity holds

$$F_k(a,P_n)=P_{(2k-1)n}. \tag{6}$$

From the recursive definition it is clear that $\deg F_k(a,t)=2k-1$ for $k\in\mathbb{N}_+$. The proof that the polynomial $F_k(a,t)$ has the property given by (6) can be easily performed by induction on k. Indeed, we note that $F_1(a,P_n)=P_n$ and the identity

$$P_n^3=\frac{a^2}{(a^2-1)^2}(P_{3n}-P_n)$$

implies that $P_{3n}=\frac{1}{a^2}P_n((a^2-1)^2P_n^2+3a^2)=F_2(a,P_n)$. We thus proved that our statement is true for $k=1,2$. Assuming now that it is true for $k-1$ and $k-2$ and using the recurrence formula, it easy (but a bit tiresome) calculation to see that our statement holds also for k. Indeed, the only thing we need to check is that the following identity

$$P_{(2k-1)n}(a)=\frac{(a^2-1)^2t^2+2a^2}{a^2}P_{2(k-1)-1}(a)-P_{2(k-2)-1}(a)$$

holds. We omit the simple details.

In order to finish the proof we need to show that for any given k, the polynomial F_k is square-free, i.e., it has not multiple roots (in a suitable field extension). To prove our result we compute the discriminant $\mathrm{Disc}(F_k(a,t))$. More precisely, we prove the formula

$$\mathrm{Disc}(F_k(a,t))=(-1)^{k+1}2^{2(k-1)}(2k-1)^{2k-1}\left(\frac{a^2-1}{a}\right)^{2(k-1)(2k-3)}.$$

Here, and in the sequel, by a discriminant or a resultant we mean discriminant and resultant with respect to the variable t.

To compute the discriminant we are interested in, we will consider the polynomial $H_k(a,t)$ instead of $F_k(a,t)$, where $H_k(a,t)=F_k(a,t)/t$. Note that the sequence $(H_k(a,t))_{k\in\mathbb{N}_+}$ satisfies the same recurrence relation as the sequence $(F_k(a,t))_{k\in\mathbb{N}_+}$. The reason that we consider the polynomials $H_k(a,t)$ is the non-vanishing of the value of $H_k(a,0)$. In fact, by a simple induction we get that $H_k(a,0)=2k-1$. It is clear that the computation of the discriminant of $F_k(a,t)$ is equivalent with the computation of the discriminant of $H_k(a,t)$. Indeed, from the identity $F_k(a,t)=tH_k(a,t)$ we get that $F_k'(a,t)=H_k(a,t)+tH_k'(a,t)$. This allow us to get the identity

$$\mathrm{Disc}(F_k(a,t))=(2k-1)^2\,\mathrm{Disc}(H_k(a,t)).$$

In the sequel we will need the following formula connecting polynomials $H_{k-1}(a,t)$, $H_k(a,t), H_k'(a,t)$. More precisely, we have

$$f_1(a,t)H_k(a,t)=f_2(a,t)H_k'(a,t)+2(2k-1)a^2H_{k-1}(a,t),$$

where

$$f_1(a,t)=2((a^2-1)^2(k-1)t^2+(2k-3)a^2),\quad f_2(a,t)=t((a^2-1)^2t^2+4a^2).$$

The above identity can be easily proved by induction on k. We are in position to compute the resultant of the polynomials $H_k(a,t), H_k'(a,t)$ and hence the discriminant of $H_k(a,t)$ via the formula

$$\mathrm{Disc}(H_k(a,t)) = (-1)^{(k-1)(2k-3)}\left(\frac{a}{a^2-1}\right)^{2(k-1)}\mathrm{Res}(H_k(a,t),H_k'(a,t)). \tag{7}$$

In order to simplify the notation we will write H_k instead of $H_k(a,t)$.
Instead of computing $\mathrm{Res}(H_k,H_k')$ we compute

$$\begin{aligned}
&\mathrm{Res}(H_k,f_2)\,\mathrm{Res}(H_k,H_k') = \mathrm{Res}(H_k,f_2H_k') && \\
&\quad= \mathrm{Res}(H_k, f_1H_k - 2(2k-1)a^2H_{k-1}) && \text{by Lemma 2.2}\\
&\quad= \left(\tfrac{a^2-1}{a}\right)^{8(k-1)}\mathrm{Res}(H_k,-2(2k-1)a^2H_{k-1}) && \text{by (4)}\\
&\quad= \left(\tfrac{a^2-1}{a}\right)^{8(k-1)}\mathrm{Res}(H_k,-2(2k-1)a^2)\,\mathrm{Res}(H_k,H_{k-1}) && \text{by (5)}\\
&\quad= \left(\tfrac{a^2-1}{a}\right)^{8(k-1)}(2(2k-1)a^2)^{2(k-1)}\,\mathrm{Res}(H_k,H_{k-1}). &&
\end{aligned}$$

We show that if $V_k = \mathrm{Res}(H_k,H_{k-1})$, then the sequence $(V_k)_{k\in\mathbb{N}_+}$ satisfies a recurrence relation

$$V_k = \left(\frac{a^2-1}{a}\right)^{4(k-1)}V_{k-1}.$$

Indeed, we have the following chain of equalities

$$\begin{aligned}
V_k &= \mathrm{Res}(H_k,H_{k-1}) = \mathrm{Res}(H_{k-1},H_k) && \text{by (3)}\\
&= \mathrm{Res}\left(H_{k-1},\tfrac{1}{a^2}((a^2-1)^2t^2+2a^2)H_{k-1}-H_{k-2}\right) &&\\
&= \left(\tfrac{a^2-1}{a}\right)^{8(k-2)}\mathrm{Res}(H_{k-1},H_{k-2}) = \left(\tfrac{a^2-1}{a}\right)^{8(k-2)}V_{k-1} && \text{by Lemma 2.2.}
\end{aligned}$$

Using the identity $V_2 = \mathrm{Res}(H_2,H_1) = 1$ we immediately get that for $k \geq 2$ we have the formula

$$V_k = \prod_{i=2}^{k}\left(\frac{a^2-1}{a}\right)^{8(i-2)} = \left(\frac{a^2-1}{a}\right)^{4(k-1)(k-2)}.$$

To finish the computation of $\mathrm{Res}(H_k,H_k')$ we need to compute the value of $\mathrm{Res}(H_k,f_2)$. The following formula can be deduced from the definition of the resultant:

$$\begin{aligned}
\mathrm{Res}(H_k,f_2) &= \mathrm{Res}(H_k,t)\,\mathrm{Res}(H_k,(a^2-1)^2t^2+4a^2)\\
&= (2k-1)\,\mathrm{Res}(H_k,(a^2-1)^2t^2+4a^2)\\
&= (2k-1)(a^2-1)^{4(k-1)}.
\end{aligned}$$

Finally, we obtain the formula for $\mathrm{Res}(H_k,H_k')$ in the following form

$$\mathrm{Res}(H_k,H_k') = \frac{\mathrm{Res}(H_k,f_2H_k')}{\mathrm{Res}(H_k,f_2)} = 2^{2(k-1)}(2k-1)^{2k-1}\left(\frac{a^2-1}{a}\right)^{4(k-1)^2}$$

and using the formula (7) we get the explicit value of $\mathrm{Disc}(H_k(a,t))$.

□

Remark 2.3. One can check by induction on k that each term of the sequence $(F_k(a,t))_{k\in\mathbb{N}_+}$ corresponds to a solution of a certain Pell type equation. More precisely, for each $k \in \mathbb{N}_+$ we have the identity

$$(2k-1)^2((a^2-1)^2F_k(a,t)^2+4a^2) = ((a^2-1)^2t^2+4a^2)F_k'(a,t)^2.$$

Our general result allow us to prove the following.

Corollary 2.4. *If $a = i\frac{\sqrt{5}-1}{2}$, where $i^2=-1$, then*

$$P_{4n-3}(a) = F_{4n-3}, P_{1-4n}(a) = F_{4n-1}$$

and the polynomial $F_k(a,t)$ has integer coefficients, and for each $k \in \mathbb{N}_+$, the Diophantine equation

$$F_k\left(i\frac{\sqrt{5}-1}{2}, t\right) = F_n \tag{8}$$

has infinitely many solutions in integers t, n.

Proof. From the Binet formula for the nth Fibonacci number we easily get the expressions for $P_{4n-3}(a)$ and $P_{1-4n}(a)$. Moreover, we observe that for $a = i\frac{\sqrt{5}-1}{2}, i^2 = -1$, we have $(a^2-1)^2/a^2 = -5$ and thus from the definition of $F_k(a,t)$ we get that our polynomial has integer coefficients. The existence of infinitely many integer solutions of the equation (8) is also clear. Indeed, from Theorem 2.1 for each $k, n \in \mathbb{N}_+$ we have the identity

$$F_k\left(i\frac{\sqrt{5}-1}{2}, (-1)^{k+1}F_{2n-1}\right) = F_{(2k-1)(2n-1)}.$$

□

Example 2.5. First few polynomials $F_k(a,(-1)^{k+1}t)$ for $a = i\frac{\sqrt{5}-1}{2}, i^2=-1$, are given in the table below.

n	$F_k(a,(-1)^{k+1}t)$
2	$t\left(5t^2-3\right)$
3	$5t\left(5t^4-5t^2+1\right)$
4	$t\left(125t^6-175t^4+70t^2-7\right)$
5	$t\left(5t^2-3\right)\left(125t^6-150t^4+45t^2-3\right)$
6	$t\left(3125t^{10}-6875t^8+5500t^6-1925t^4+275t^2-11\right)$
7	$t\left(15625t^{12}-40625t^{10}+40625t^8-19500t^6+4550t^4-455t^2+13\right)$

Table 1. The polynomials $F_k(a,(-1)^{k+1}t)$ for $a = i\frac{\sqrt{5}-1}{2}, i^2=-1$, and $k = 2,\ldots,7$.

Remark 2.6. It is not difficult to observe that for the sequence $Q_n = Q_n(a) = a^n + a^{-n}$, where $a \in \mathbb{C}\setminus\{-1,0,1\}$, one can construct a polynomial $G_k \in \mathbb{Z}[t]$ of degree k such that the equation $G_k(x) = Q_m(a)$ has infinitely many solutions in $x, m \in \mathbb{N}_+$. Indeed, in order to do see that it is enough to observe that the following (easily to establish) identity holds:

$$Q_{kn}(a) = Q_n(a)Q_{(k-1)n}(a) - Q_{(k-2)n}(a).$$

Thus, if we define $G_1(t) = t, G_2(t) = t^2-2$ and $G_k(t) = tG_{k-1}(t) - G_{k-2}(t)$ for $k \geq 2$, then we have $G_k(Q_n) = Q_{kn}$ and hence the result. Moreover, it is not difficult to show that the polynomial G_k is square-free. Indeed, essentially the same type of reasoning as presented in the proof of second part of Theorem 2.1 can be used for the computation of $\mathrm{Disc}(G_k(t))$. Indeed, the only non-obvious fact we need to know is the existence of the formula connecting G_k, G_k', G_{k-1}. The mentioned formula takes the form

$$ktG_k(t) = (t^2-4)G_k'(t) + 2kG_{k-1}(t),$$

and the rest of the proof goes exactly in the same way as in the case of $F_k(a,t)$. As a final result we get the formula $\mathrm{Disc}(G_k(t)) = 2^{k-1}k^k$. Moreover, it is easy to prove (by induction on k) that the following identity holds:

$$(k+1)^2(G_k(t)^2-4) = (t^2-4)G_k'(t)^2.$$

We omit the details.

In particular, if $a = \frac{\sqrt{5}-1}{2}$, then one can easily check that $Q_{2n}(a) = L_{2n}$, where L_n is nth Lucas number. Thus, as consequence, we get that there is a polynomial $G_k \in \mathbb{Z}[t]$ of degree $k+1$ such that the Diophantine equation $G_k(t) = L_m$ has infinitely many solutions in integers k, m. Indeed, it is enough to note the identity

$$G_k(L_{2n}) = L_{2kn}.$$

3. Numerical and experimental results

In light of Theorem 2.1 it is natural to ask about construction of polynomials of even degree, say d, which represent "many" Fibonacci numbers. We are very modest here and asks about the existence of polynomials $f \in \mathbb{Q}[x]$ of degree d such that the Diophantine equation $f(x) = F_m$ has at least $d+2$ solutions. We are especially interested in the case $d = 2$.

To find interesting examples we performed the following search strategy. We first generated the set

$$A = \{(F_p, F_q, F_r, F_s):\ p, q, r, s \in \{2, \ldots, 100\}\}$$

and then for each quadruple with pairwise distinct elements $v \in A$ (there are exactly 3764376 elements of this kind in A) we looked for the degree two polynomial f such that

$$f(1) = F_p, f(2) = F_q, f(3) = F_r, f(4) = F_s. \tag{9}$$

Note that a degree two polynomial is defined by three coefficients and thus our system of equation (9) is over-determined. Thus, we cannot expect too many solutions (if any). In fact, with this approach we found 93 polynomials with required properties. Browsing through the set of solutions we were able to find three infinite families $(f_{i,n})_{n\in\mathbb{N}_+}, i = 1, 2, 3,$ of degree two polynomials satisfying required conditions. More precisely, we define

$$\begin{aligned}
f_{1,n}(x) &= (F_{2n+1}x - L_{2n})((3F_{2n+1} - F_{2n} - F_{2n-3})x - 2F_{2n} - 5F_{2n-1} + F_{2n-5}),\\
f_{2,n}(x) &= F_{2n+3}x^2 - (3F_{2n+3} - F_{2n})x + 2F_{2n+3} - F_{2n-2},\\
f_{3,n}(x) &= (F_{2n}x - F_{2n+1} + F_{2n-3})((F_{2n+2} - F_{2n-2})x - 5F_{2n-1}).
\end{aligned}$$

With $f_{i,n}$ defined above it is easy to check that the following equalities are true:

$$\begin{array}{llll}
f_{1,n}(1) = F_{4n-2} & f_{1,n}(2) = F_{4n} & f_{1,n}(3) = F_{4n+4} & f_{1,n}(4) = F_{4n+6},\\
f_{2,n}(1) = F_{2n-1} & f_{2,n}(2) = F_{2n+1} & f_{2,n}(3) = F_{2n+5} & f_{2,n}(4) = F_{2n+7},\\
f_{3,n}(1) = F_{4n-4} & f_{3,n}(2) = F_{4n-2} & f_{3,n}(3) = F_{4n+2} & f_{3,n}(4) = F_{4n+4}.
\end{array}$$

Note that from the result of Nemes and Pethő we know that for each $n \in \mathbb{N}_+$ and $i \in \{1,2,3\}$ the Diophantine equation $f_{i,n}(x) = F_m$ has only finitely many solutions in integers.

It should be noted that among 93 polynomials found by the above described search, there is only one which do not belong to the sequences $(f_{i,n})_{n\in\mathbb{N}}, i \in \{1,2,3\}$. This sporadic polynomial is the following:

$$f(x) = \frac{1}{2}(x^2 - x + 4). \tag{10}$$

Unexpectedly, it represents five Fibonacci numbers. More precisely, all non-negative integer solutions (x, m) of the Diophantine equation $f(x) = F_m$ are

$$(x, m) = (0, 3), (1, 3), (2, 4), (3, 5), (4, 6), (22, 13).$$

For the proof of this result see Theorem 4.1 below. Let us also note that $f(x) = t_{x-1} + 2$, where $t_x = x(x + 1)/2$ is the xth triangular number. Thus the problem of finding non-negative solutions of Diophantine equation $f(x) = F_m$ is equivalent with the finding triangular numbers of the form $F_m - 2$.

We gather our experimental data into the following general observation.

Conjecture 3.1. *Let $i \in \{1,2,3\}$. If $i = 1$ or $(i, n) \neq (2, 2), (3, 1)$, then the only integer solutions (t, n, m) of the Diophantine equation $f_{i,n}(t) = F_m$ correspond to $t = 1, 2, 3, 4$.*

If $(i, n) = (2, 2)$, then the Diophantine equation $f_{2,2}(t) = F_m$ has exactly five integer solutions with $m \geq 0$ given by $(t, m) = (-4, 14), (1, 3), (2, 5), (3, 9), (4, 11)$.

If $(i, n) = (3, 1)$, then the Diophantine equation $f_{3,1}(t) = F_m$ has exactly seven integer solutions with $m \geq 0$ given by $(t, m) = (-36, 19), (0, 5), (1, 0), (2, 1), (2, 2), (3, 6), (4, 8)$.

We performed similar analysis in case of Lucas numbers and were able to spot one infinite family $(g_n)_{n\in\mathbb{N}_{\geq 4}}$, where

$$g_n(x) = L_n x^2 - (L_{n+2} + L_{n-4})x + 5L_{n-2}.$$

Then

$$g_n(1) = L_{n-4}, \quad g_n(2) = L_{n-2}, \quad g_n(3) = L_{n+2}, \quad g_n(4) = L_{n+4}.$$

Remarkably, if $g \in \mathbb{Z}[x]$ is of degree 2 and satisfies

$$g(1) = L_p, g(2) = L_q, g(3) = L_r, g(4) = L_s$$

for $p < q < r < s \leq 100$ then there is $n \leq 45$ satisfying $g(x) = g_n(x)$.

Conjecture 3.2. *If $n \neq 2$, then the only intger solutions (t, n, m) of the Diophantine equation $g_n(t) = L_m$ correspond to $t = 1, 2, 3, 4$.*

If $n = 2$, then the Diophantine equation $g_2(t) = L_m$ has exactly five integer solutions with $m \geq 0$ given by $(t, m) = (1, 2), (2, 0), (3, 4), (4, 6), (12, 12)$.

We performed similar search in the case of degree 4 polynomials. More precisely, we were interested in finding examples of polynomials $f \in \mathbb{Q}[x]$ of degree 4 such that f represents at least 6 Fibonacci numbers. We first generated the set

$$B = \{(F_p, F_q, F_r, F_s, F_u, F_v) : \ p, q, r, s, u, v \in \{2, \ldots, 60\}\}$$

and then for each sextuple with pairwise distinct elements $w \in B$ (there are exactly 45057474 elements of this kind in B) we looked for a polynomial f of degree ≤ 4 such that

(11)
$$f(1) = F_p, \quad f(2) = F_q, \quad f(3) = F_r, \quad f(4) = F_s, \quad f(5) = F_u, \quad f(6) = F_v.$$

In the considered range we found only four polynomials of degree ≤ 4 representing at least six Fibonacci polynomials. They are given in the able below together with the known integer solutions of the equation $f(x) = F_m, m \geq 0$.

$f(t)$	Known solutions of $f(x) = F_m$
$\left(t^3 - 6t^2 + 23t - 12\right)/6$	$(1,1), (1,2), (2,4), (3,5),$ $(4,6), (5,7), (6,8)$
$\left(101t^4 - 1064t^3 + 4369t^2 - 7162t + 3780\right)/12$	$(1,3), (2,4), (3,11),$ $(4,13), (5,15), (6,17)$
$\left(245t^4 - 3080t^3 + 13729t^2 - 23290t + 12420\right)/12$	$(1,3), (2,4), (3,13)$ $(4,14), (5,15), (6,17)$
$\left(t^4 - 6t^3 + 35t^2 - 6t + 96\right)/24$	$(-4,10), (-2,7), (1,5), (2,6),$ $(3,7), (4,8), (5,9), (6,10)$

Table 2. The polynomials $f \in \mathbb{Q}[x]$ of degree $3 \leq d \leq 4$ such that $f(x)$ represents at least six Fibonacci numbers together with the set of known integral solutions of the corresponding Diophantine equation $f(x) = F_m$.

We note that in case of the polynomial $f(x) = (x^3 - 6x^2 + 23x - 12)/6$ the Diophantine equation $f(x) = F_m$ can be reduced to genus 2 curves by using the identity $L_n^2 = 5F_n^2 \pm 4$. Therefore one may try to apply the method developed in [7] to determine a complete list of integral solutions. The hyperelliptic curves $y^2 = 5f(x)^2 \pm 4$ define genus 2 curves and their Jacobians have rank 4. It turns out to be difficult to provide generators of the Mordell-Weil groups that is required to apply the method based on Baker's linear forms in logarithms and the so-called Mordell-Weil sieve. We finish this section with the following.

Conjecture 3.3. *Let f be a polynomial from* Table 2. *There are no other solutions of the Diophantine equation $f(x) = F_m$ than those presented in the right column.*

Remark 3.4. We note that for any given even $d \in \mathbb{N}_+$ it is possible to construct a polynomial $G_d \in \mathbb{Q}[x]$ of degree d such that the equation $G_d(x) = F_m$ has at least $d + 2$ solutions in integers x, m. Indeed, we learned that if we define G_d as the unique polynomial of degree d with rational coefficients satisfying the system of equations

$$G_d(0) = F_{d+2},\ G_d(1) = F_{d+3},\ \ldots,\ G_d(d-1) = F_{2d},\ G_d(d) = F_{2d+2},$$

then additional the equality $G_d(-1) = F_{d+1}$ holds [15, Equation (3.1)].

The paucity of even degree polynomials $d \geq 4$ representing at least $d+2$ Fibonacci numbers suggest the following general problem.

Problem 3.5. *For any given even $d \geq 4$ construct infinitely many (non-trivial) polynomials of degree d representing at least $d + 2$ Fibonacci polynomials.*

Let $k, d \in \mathbb{N}_{\geq 2}$ and define the following set

$$\begin{aligned} A(k,d) =\{&f \in \mathbb{Q}[x] :\ \deg f = d \text{ and the equation } f(x) = F_m \\ &\text{has finitely many and at least } k \text{ integer solutions } (x,m)\}, \end{aligned}$$

the condition

$$B(k,d): \text{there are infinitely many} f \in \mathbb{Q}[x] \text{ with } \deg f = d \text{ and the equation}$$
$$f(x) = F_m \text{ has finitely many and at least } k \text{ integer solutions,}$$

and the corresponding quantity

$$m(d) := \max\{\sharp A(k,d) : \ k \in \mathbb{N}_{\geq d+2}\}.$$

Problem 3.6. *Let $d \in \mathbb{N}_{\geq 2}$.*

(1) *Show that $m(d) = O(d)$.*
(2) *For which values of $k \geq d+2$ is the condition $B(k,d)$ true?*

We know that $m(d) \geq d+2$. Moreover, our findings presented in Section 3 shows that the condition $B(4,2)$ is true.

4. Fibonacci numbers represented by shifted triangular numbers and the equation $F_n = \binom{X}{5}$

Motivated by the example given by (10) we deal with the family of equations

$$t_{x-1} + d = \binom{x}{2} + d = F_n \quad \text{for } -20 \leq d \leq 20. \tag{12}$$

Theorem 4.1. *All non-negative integral solutions n with $-20 \leq d \leq 20$ of equation (12) are as follows*

$d=-20, n \in \{1,2,6,13,15\}, d=-19, n \in \{3\}, d=-18, n \in \{4\}, d=-16, n \in \{5,11\},$
$d=-15, n \in \{0,7,8\}, d=-14, n \in \{1,2\}, d=-13, n \in \{3,6\}, d=-12, n \in \{4\},$
$d=-11, n \in \{9,10\}, d=-10, n \in \{0,5\}, d=-9, n \in \{1,2,12\}, d=-8, n \in \{3,7\},$
$d=-7, n \in \{4,6,8\}, d=-6, n \in \{0\}, d=-5, n \in \{1,2,5,19\}, d=-4, n \in \{3\},$
$d=-3, n \in \{0,4,16\}, d=-2, n \in \{1,2,6,7,9,11\}, d=-1, n \in \{0,3,5,14\},$
$d=0, n \in \{0,1,2,4,8,10\}, d=1, n \in \{1,2,3,17\}, d=2, n \in \{3,4,5,6,13\},$
$d=3, n \in \{4,7\}, d=4, n \in \{5\}, d=5, n \in \{5,6\}, d=6, n \in \{8,9\}, d=7, n \in \{6,7\},$
$d=8, n \in \{6,12,24\}, d=10, n \in \{7,10\}, d=11, n \in \{8,11\}, d=12, n \in \{7\},$
$d=13, n \in \{7,9\}, d=15, n \in \{8,15\}, d=18, n \in \{8\}, d=19, n \in \{9,10\}, d=20, n \in \{8\}.$

Proof. We use the following well-known identity related to the sequences F_n and L_n

$$L_n^2 - 5F_n^2 = 4(-1)^n. \tag{13}$$

The above identity yields the hyperelliptic curves

$$C_{d,\pm}: \quad y^2 = 5x^4 - 10x^3 + (20d+5)x^2 - 20dx + 20d^2 \pm 16.$$

We searched for small solutions on these curves. If no points were found, then we used the Magma procedure `TwoCoverDescent()` [5] to show that there exist no solutions. In case that certain small solutions exist we used the Magma procedure `IntegralQuarticPoints()` based on results obtained by Tzanakis [30]. As an example consider the case $d = -5$. Since $(0, -22)$ is a point on the curve $C_{-5,-}$ we used `IntegralQuarticPoints([ 5, -10, -95, 100, 484 ],[0,-22])` to determine a complete list of integral solutions. It turns out that there are solutions only if

$$x \in \{-91, -4, -3, -2, 0, 1, 3, 4, 5, 92\}.$$

Similarly, $(-3,-6)$ is a point on $C_{-5,+}$. Therefore via `IntegralQuarticPoints([ 5, -10, -95, 100, 516 ],[-3,-6])` it follows that

$$x \in \{-3, 4\}.$$

Thus we have the solutions

$$\begin{aligned} \binom{4}{2} - 5 &= F_1 = F_2, \\ \binom{5}{2} - 5 &= F_5, \\ \binom{92}{2} - 5 &= F_{19}. \end{aligned}$$

□

Let us consider the Diophantine equation

$$F_n = \binom{X}{5} \tag{14}$$

with $X \geq 5$. We have the following result.

Theorem 4.2. *The integral solutions of equation* (14) *with* $X \geq 5$ *are given by* $(n, X) \in \{(1, 5), (2, 5), (8, 7)\}$.

Proof. In case of equation (14) by applying the identity (13) we obtain

$$5\binom{X}{5}^2 \pm 4 = L_n^2.$$

Hence we need to compute the integral solutions of the equations

$$C_\delta : \qquad y^2 = x^2(x + 15)^2(x + 20) + 4 \cdot 5^4 \cdot (5!)^2 \cdot \delta,$$

where $\delta \in \{-1, 1\}$ and $x = 5X^2 - 20X$. That is we deal with genus 2 curves. By using Magma [4] we can determine generators of the Mordell-Weil groups based on Stoll's papers [24], [25], [26]. Let us denote the Jacobians of the curves C_δ by J_δ, where $\delta \in \{-1, 1\}$. We get that J_{-1} is free of rank 2 with Mordell-Weil basis given by (in Mumford representation)

$$\begin{aligned} d_1 &=< x - 25, 3000 >, \\ d_2 &=< x^2 + 75x + 1500, 600x + 24000 > \end{aligned}$$

and J_1 is free of rank 4 with Mordell-Weil basis given by

$$\begin{aligned} D_1 &=< x, 6000 >, \\ D_2 &=< x + 40, 4000 >, \\ D_3 &=< x^2 + 15x, 6000 >, \\ D_4 &=< x^2 + 15x - 1000, 200x + 4000 > . \end{aligned}$$

Baker's method [2] can be applied to get large upper bounds B_δ for $\log |x|$. Using the improvements given in [7] and [14] we obtain that

$$B_{-1} = 2.26 \cdot 10^{493} \text{ and } B_1 = 1.11 \cdot 10^{503}.$$

Every integral point on the curves can be expressed in the forms

$$P-\infty=\sum_{i=1}^{2} m_i d_i \text{ and } P-\infty=\sum_{i=1}^{4} n_i D_i,$$

where m_1, m_2, n_1, n_2, n_3 and n_4 are integers. According to Proposition 6.2 in [14] we compute the period matrix and the hyperelliptic logarithms with 1200 digits of precision in case of both curves. The hyperelliptic logarithms of the divisors d_i are as follows

$$\begin{aligned}
\varphi(d_1) &= (-0.018478\ldots + i0.009553\ldots, -0.397546\ldots + i0.372090\ldots) \in \mathbb{C}^2,\\
\varphi(d_2) &= (0.020606\ldots - i0.005882\ldots, -0.861905\ldots + i0.814915\ldots) \in \mathbb{C}^2.
\end{aligned}$$

In case of the rank 4 curve we obtain

$$\begin{aligned}
\varphi(D_1) &= (-0.020382\ldots + i0.004844\ldots, -1.182385\ldots - i0.446046\ldots) \in \mathbb{C}^2,\\
\varphi(D_2) &= (-0.013432\ldots - i0.004844\ldots, -1.326128\ldots - i0.446046\ldots) \in \mathbb{C}^2,\\
\varphi(D_3) &= (-0.011009\ldots + i0.004844\ldots, -0.854126\ldots - i0.446046\ldots) \in \mathbb{C}^2,\\
\varphi(D_4) &= (-0.007101\ldots - i0.004844\ldots, -1.160439\ldots - i0.446046\ldots) \in \mathbb{C}^2.
\end{aligned}$$

Based on Proposition 6.2 in [14] we set $K := 10^{1000}$ for both curves and the reductions yield that

$$||(m_1, m_2)|| \leq 45.65 \quad \text{and} \quad ||(n_1, n_2, n_3, n_4)|| \leq 103.27.$$

Repeat reductions with $K := 10^{20}, 10^{14}, 10^{12}$ provide the following bounds

$$||(m_1, m_2)|| \leq 6.36 \quad \text{and} \quad ||(n_1, n_2, n_3, n_4)|| \leq 13.73.$$

Enumeration of possible linear combinations up to these bounds provide that

$$x \in \{-40, -20, -15, 0, 25, 105, 1425/4\}.$$

It remains to determine the corresponding values of X, these are as follows

$$X \in \{-3, -1, 0, 1, 2, 3, 4, 5, 7\}.$$

Therefore $X = 5, n = 1, 2$ and $X = 7, n = 8$ are the only non-trivial solutions. □

Acknowledgement. The authors express their gratitude to the anonymous referee for constructive suggestions which improved the quality of the paper. The research of the first author was partially supported in part by grants ANN130909, K115479 and K128088 of the Hungarian National Foundation for Scientific Research. The research of the second author was partially supported by the grant of the Polish National Science Centre no. UMO-2019/34/E/ST1/00094

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Szabolcs Tengely
Mathematical Institute
University of Debrecen
P.O.Box 12
4010 Debrecen
Hungary
Email address: tengely@science.unideb.hu

Maciej Ulas
Jagiellonian University
Faculty of Mathematics and Computer Science
Institute of Mathematics
Łojasiewicza 6
30-348 Kraków
Poland
Email address: Maciej.Ulas@im.uj.edu.pl