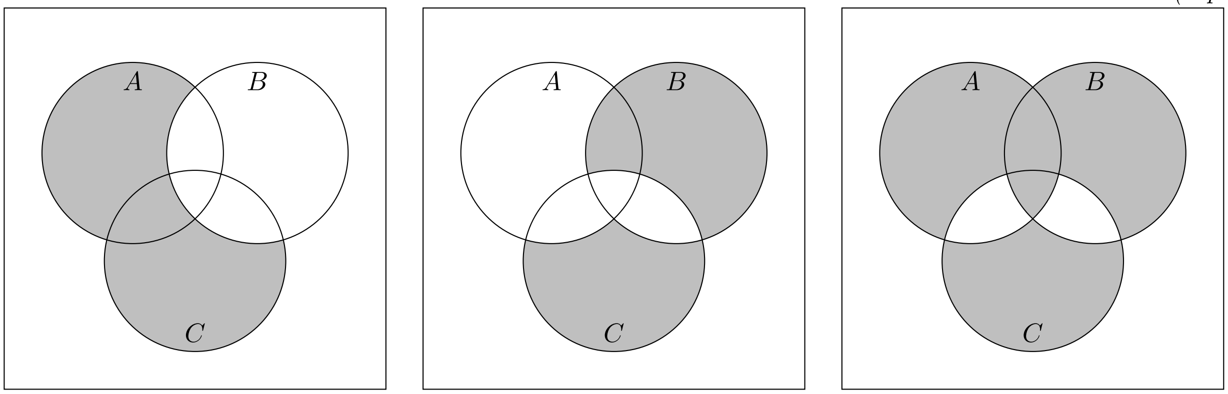
Discrete Mathematics

November 2025

Exercise 1. *Use set notation to describe the shaded areas:* *(2 point)*



Exercise 2. *Draw a Venn diagram for the following sets:*
$A \cup B \cup C = \{6, 8, 9, 10, 12, 14, 15, 30\}$
A *contains even numbers,*
B *contains numbers divisible by 3,*
C *contains numbers divisible by 5.* *(1 point)*

Exercise 3. *Provide three sets* A, B *and* C *which satisfy the following cardinality conditions*

$$|A \cap B \cap C| = 1,$$
$$|A \cap B| = 2, \quad |A \cap C| = 3, \quad |B \cap C| = 3,$$
$$|A| = 5, \quad |B| = 6, \quad |C| = 6.$$

(1 point)

Exercise 4. *(a) How many solutions does the equation* $x_1 + x_2 + x_3 + x_4 = 3$ *have, where* x_1, x_2, x_3, x_4 *are integers such that* $x_i \geq -i + 2$?

(b) How many solutions does the equation $x_1 + x_2 + x_3 + x_4 + x_5 = 3$ *have, where* x_1, x_2, x_3, x_4, x_5 *are integers such that* $x_i \geq -i - 1$? *(2 point)*

Exercise 5. *(a) How many eight digit numbers can be formed from the digits 1,1,1,2,2,2,1,1?*
(b) How many seven digit numbers can be formed from the digits 0,1,1,1,2,2,1? *(2 points)*

Exercise 6. *Use the Euclidean algorithm to find* $\gcd(a, b)$ *and compute integers* x *and* y *for which*

$$ax + by = \gcd(a, b):$$

$a = 1501, b = 1007.$ *(2 points)*

Exercise 7. *a. Determine the decimal representation of the following numbers.*

$2230_5 =$ .... $_{10}$

$2230_6 =$ .... $_{10}$

$2230_7 =$ .... $_{10}$ *(2 points)*

b. Determine the appropriate representations of the following numbers.

$2123_{10} =$ .... $_{3}$

$2123_{10} =$ .... $_{5}$

$2123_{10} =$ .... $_{8}$ *(2 points)*

Exercise 8. *Expand the following expressions using the binomial theorem.*

(1) $(-xy + 2x - 3y)^2$,

(2) $\left(2x - \frac{x}{2y}\right)^3$.

(2 points)

Exercise 9. *Prove that* $n^2 - 1$ *is divisible by 8 for all odd positive integers* n. *(2 points)*

Exercise 10. *Prove that $17n^3 + 103n$ is divisible by 6 for all positive integers n.* *(2 points)*

Exercise 11. *Find the value of k for which $k\binom{12}{k}$ is largest.* *(2 points)*

Exercise 12. *Describe all values of n and k for which*

$$\binom{n}{k+1} = 12\binom{n}{k}.$$

(2 points)

Exercise 13. *Suppose a_n is a sequence such that $a_{n+2} = a_{n+1} - a_n$ for all $n \geq 1$. Given that $a_{19} = 5$ and $a_{23} = -1$, find a_1.* *(2 points)*

Exercise 14. *What is the solution of the recurrence relation $a_n = a_{n-1} + 2a_{n-2}, n \geq 2$ and $a_0 = 2, a_1 = 7$? (3 points)*

Exercise 15. *Let $a_0 = -2, a_1 = 1$ and $a_n = 3a_{n-1} - 2a_{n-2}$ if $n \geq 2$. Prove that $a_n = 3 \cdot 2^n - 5$.* *(3 points)*

Exercise 16. *Find a closed formula for a_n, where* *(3 points)*

$$\begin{aligned} a_0 &= 3, \\ a_1 &= 0, \\ a_2 &= 14, \\ a_n &= 7a_{n-2} + 6a_{n-3}, \qquad \text{if } n \geq 3. \end{aligned}$$

Exercise 17. *Define a sequence $a_1, a_2, \ldots$ by $a_1 = 5/2$ and $a_{n+1} = a_n^2 - 2$ for $n \geq 1$. Give an explicit formula for a_n and prove it.* *(3 points)*

Exercise 18. *Provide a bound N for n such that the equation is solvable in non-negative integers if $n \geq N$:*

$$13x + 19y = n.$$

(3 points)

Exercise 19. *Let the sequence T_n defined by $T_1 = T_2 = T_3 = 1$ and $T_n = T_{n-1} + T_{n-2} + T_{n-3}$ for $n \geq 4$. Prove that*

$$T_n < 2^n, \quad n \in \mathbb{N}.$$

(3 points)

Exercise 20. *Let $a_n = m + dn$ for some integers m, d. Given that*

$$\frac{a_i - a_j}{a_k} + \frac{a_k - a_i}{a_j} + \frac{a_j - a_k}{a_i} = 0,$$

compute the possible values of d. *(3 points)*

Exercise 21. *Prove that if k is odd, then 2^{n+2} divides*

$$k^{2^n} - 1$$

for all natural numbers n. *(3 points)*

Exercise 22. *Let F_n be the sequence of Fibonacci numbers. Show that for all positive integers m*

$$\sum_{j=1}^{m} F_j F_{j+1} = F_{m+1}^2 - \frac{1}{2}(1 + (-1)^m).$$

(3 points)

Exercise 23. *Let F_n be the sequence of Fibonacci numbers. Show that for all positive integers m*

$$\sum_{j=1}^{m} \frac{F_{j-1}}{2^j} = 1 - \frac{F_{m+2}}{2^m}.$$

(3 points)