
MZFK

Diophantine Equations Related to Arithmetic Progressions

Szabolcs Tengely

May 14, 2007

- $x^2 + q^m = 2^r y^p$ (joint work with Samir Siksek)

- $x^2 + q^m = 2^r y^p$ (joint work with Samir Siksek)
  - Special cases
  - Modular approach

- $x^2 + q^m = 2^r y^p$ (joint work with Samir Siksek)
  - Special cases
  - Modular approach
- $n(n+d)\cdots(n+(k-1)d) = by^m$

Outline

- $x^2 + q^m = 2^r y^p$ (joint work with Samir Siksek)
  - Special cases
  - Modular approach
- $n(n+d)\cdots(n+(k-1)d) = by^m$
  - $m = 2$, the cases $k = 5, 7$ (lecture by Rob Tijdeman and lecture by Shanta Laisram)
  - $m = 3$, lecture by Lajos Hajdu

- $x^2 + q^m = 2^r y^p$ (joint work with Samir Siksek)
  - Special cases
  - Modular approach
- $n(n+d)\cdots(n+(k-1)d) = by^m$
  - $m = 2$, the cases $k = 5, 7$ (lecture by Rob Tijdeman and lecture by Shanta Laisram)
  - $m = 3$, lecture by Lajos Hajdu
- squares and cubes in arithmetic progressions, paper by Nils Bruin, Kálmán Győry, Lajos Hajdu, Szabolcs Tengely

- BHV works in many cases

- BHV works in many cases
  - $x^2 + 2^a 3^b = y^p$ Luca (2002)
  - $x^2 + p^{2k+1} = 4y^n$ Arif and Al-Ali (2002)
  - $x^2 + 5^{2k} = y^n$ Muriefah (2006)

- BHV works in many cases
  - $x^2 + 2^a 3^b = y^p$ Luca (2002)
  - $x^2 + p^{2k+1} = 4y^n$ Arif and Al-Ali (2002)
  - $x^2 + 5^{2k} = y^n$ Muriefah (2006)
- $p = 3$, S-integral points on elliptic curves

$$\left(\frac{x}{q^{3t}}\right)^2 + q^s = 2^r \left(\frac{y}{q^{2t}}\right)^3$$

Special cases

- BHV works in many cases
  - $x^2 + 2^a 3^b = y^p$ Luca (2002)
  - $x^2 + p^{2k+1} = 4y^n$ Arif and Al-Ali (2002)
  - $x^2 + 5^{2k} = y^n$ Muriefah (2006)
- $p = 3$, S-integral points on elliptic curves

$$\left(\frac{x}{q^{3t}}\right)^2 + q^s = 2^r \left(\frac{y}{q^{2t}}\right)^3$$

- $p = 5$, algebraic number theory, Thue-equations (lecture by Yann Bugeaud)

Lucas sequence: $u_n(\alpha,\beta)=\frac{\alpha^n-\beta^n}{\alpha-\beta}$, p is a primitive divisor of $u_n(\alpha,\beta)$ if p divides u_n , but does not divide $(\alpha-\beta)^2u_1u_2\cdots u_{n-1}$.

Arif and Al-Ali (2002):

$$x^2+3^{2k+1}=4y^p$$

We obtain

$$\frac{x+3^k\sqrt{-3}}{2}=\left(\frac{a+b\sqrt{-3}}{2}\right)^p.$$

Let $\alpha=\frac{a+b\sqrt{-3}}{2}$, $\beta=\frac{a-b\sqrt{-3}}{2}$. We have

$$u_n(\alpha,\beta)=\begin{cases}\pm 1 \text{ if } p\neq 3,\\ \pm 3 \text{ if } p=3.\end{cases}$$

$$x^2 + 5^k = 2y^3,$$

here $S = \{5\}$,

$$\left(\frac{2x}{5^{3t}}\right)^2 = \left(\frac{2y}{5^{2t}}\right)^3 - 4 \cdot 5^s, \quad s \in \{0, 1, \dots, 5\}$$

using MAGMA one obtains all the S-integral points on the curve. The solutions of the original problem:

$$(x, y) \in \{(\pm 1, 1), (\pm 7, 3), (\pm 99, 17)\}.$$

$x^2 + 3^{2m} = 2y^3$, factor the LHS $3^m = (u - v)(u^2 + 4uv + v^2)$, hence there exists $k \in \{0, \ldots, m\}$ such that

$x^2 + 3^{2m} = 2y^3$, factor the LHS $3^m = (u-v)(u^2+4uv+v^2)$, hence there exists $k \in \{0, \dots, m\}$ such that

$$\begin{aligned} u - v &= \pm 3^k, \\ u^2 + 4uv + v^2 &= \pm 3^{m-k}. \end{aligned}$$

$x^2 + 3^{2m} = 2y^3$, factor the LHS $3^m = (u-v)(u^2+4uv+v^2)$, hence there exists $k \in \{0, \ldots, m\}$ such that

$$\begin{aligned} u - v &= \pm 3^k, \\ u^2 + 4uv + v^2 &= \pm 3^{m-k}. \end{aligned}$$

That is

$$6v^2 \pm 6(3^k)v + 3^{2k} = \pm 3^{m-k}.$$

If $k = 0$ or $k = m$, then $(x, y) = (\pm 1, 1)$.
If $k = m - 1 > 0$, then $3 \mid 2v^2 \pm 1$.

$x^2 + 3^{2m} = 2y^3$, factor the LHS $3^m = (u-v)(u^2+4uv+v^2)$, hence there exists $k \in \{0, \dots, m\}$ such that

$$\begin{aligned} u - v &= \pm 3^k, \\ u^2 + 4uv + v^2 &= \pm 3^{m-k}. \end{aligned}$$

That is

$$6v^2 \pm 6(3^k)v + 3^{2k} = \pm 3^{m-k}.$$

If $k = 0$ or $k = m$, then $(x, y) = (\pm 1, 1)$.
If $k = m - 1 > 0$, then $3 \mid 2v^2 \pm 1$.

$$\begin{aligned} u - v &= -3^{m-1}, \\ u^2 + 4uv + v^2 &= -3. \end{aligned}$$

$$u = \frac{-\varepsilon}{2}\left((2+\sqrt{3})^{t-1} + (2-\sqrt{3})^{t-1}\right),$$
$$v = \frac{\varepsilon}{2}\left((2+\sqrt{3})^{t} + (2-\sqrt{3})^{t}\right),$$

where $t \in \mathbb{N}, \varepsilon \in \{-1, 1\}$.

$$u = \frac{-\varepsilon}{2}\left((2+\sqrt{3})^{t-1} + (2-\sqrt{3})^{t-1}\right),$$
$$v = \frac{\varepsilon}{2}\left((2+\sqrt{3})^{t} + (2-\sqrt{3})^{t}\right),$$

where $t \in \mathbb{N}, \varepsilon \in \{-1, 1\}$.

$$\frac{1}{2}\left((3+\sqrt{3})(2+\sqrt{3})^{t-1} + (3-\sqrt{3})(2-\sqrt{3})^{t-1}\right) = \pm 3^{m-1}.$$

$$u = \frac{-\varepsilon}{2}\left((2+\sqrt{3})^{t-1} + (2-\sqrt{3})^{t-1}\right),$$
$$v = \frac{\varepsilon}{2}\left((2+\sqrt{3})^{t} + (2-\sqrt{3})^{t}\right),$$

where $t \in \mathbb{N}, \varepsilon \in \{-1, 1\}$.

$$\frac{1}{2}\left((3+\sqrt{3})(2+\sqrt{3})^{t-1} + (3-\sqrt{3})(2-\sqrt{3})^{t-1}\right) = \pm 3^{m-1}.$$

Recurrence sequence $r_0 = r_1 = 3, r_t = 4r_{t-1} - r_{t-2}, t \geq 2.$

$$u = \frac{-\varepsilon}{2}\left((2+\sqrt{3})^{t-1} + (2-\sqrt{3})^{t-1}\right),$$
$$v = \frac{\varepsilon}{2}\left((2+\sqrt{3})^{t} + (2-\sqrt{3})^{t}\right),$$

where $t \in \mathbb{N}, \varepsilon \in \{-1, 1\}$.

$$\frac{1}{2}\left((3+\sqrt{3})(2+\sqrt{3})^{t-1} + (3-\sqrt{3})(2-\sqrt{3})^{t-1}\right) = \pm 3^{m-1}.$$

Recurrence sequence $r_0 = r_1 = 3, r_t = 4r_{t-1} - r_{t-2}, t \geq 2$.

$$r_t \equiv 0 \pmod{27} \iff t \equiv 5 \text{ or } 14 \pmod{18},$$
$$r_t \equiv 0 \pmod{17} \iff t \equiv 5 \text{ or } 14 \pmod{18}.$$

$$u = \frac{-\varepsilon}{2}\left((2+\sqrt{3})^{t-1} + (2-\sqrt{3})^{t-1}\right),$$
$$v = \frac{\varepsilon}{2}\left((2+\sqrt{3})^{t} + (2-\sqrt{3})^{t}\right),$$

where $t \in \mathbb{N}, \varepsilon \in \{-1, 1\}$.

$$\frac{1}{2}\left((3+\sqrt{3})(2+\sqrt{3})^{t-1} + (3-\sqrt{3})(2-\sqrt{3})^{t-1}\right) = \pm 3^{m-1}.$$

Recurrence sequence $r_0 = r_1 = 3, r_t = 4r_{t-1} - r_{t-2}, t \geq 2$.

$$r_t \equiv 0 \pmod{27} \iff t \equiv 5 \text{ or } 14 \pmod{18},$$
$$r_t \equiv 0 \pmod{17} \iff t \equiv 5 \text{ or } 14 \pmod{18}.$$

There are two possible cases: $m = 2, k = 1 : (x, y) = (13, 5)$, and $m = 3, k = 2 : (x, y) = (545, 53)$.

If $r > 1$ then $x^2 + 13^m \equiv 2 \pmod 4$ and $2^r y^p \equiv 0 \pmod 4$, a contradiction.

$$r = 0: \quad x^2 + 13^m = y^p, \quad x \text{ is even }, y \text{ odd.}$$
$$r = 1: \quad x^2 + 13^m = 2y^p, \quad x \text{ is odd }, y \text{ is odd.}$$

If $r > 1$ then $x^2 + 13^m \equiv 2 \pmod 4$ and $2^r y^p \equiv 0 \pmod 4$, a contradiction.

$$r = 0: \quad x^2 + 13^m = y^p, \quad x \text{ is even }, y \text{ odd.}$$
$$r = 1: \quad x^2 + 13^m = 2y^p, \quad x \text{ is odd }, y \text{ is odd.}$$

The case $r = 0$. We have the following two Frey curves (Ivorra and Kraus (2006), lecture by Ivorra)

$$E_1: \quad Y^2 = X^3 + 2xX^2 + y^p X,$$
$$E_2: \quad Y^2 = X^3 + 2xX^2 + (x^2 - y^p)X.$$

By Ribet's level-lowering one gets

$$N_p(E_1) = 2^s \cdot 13 \text{ where } s = \begin{cases} 6 \text{ if } x \equiv 1 \pmod 4, \\ 5 \text{ if } x \equiv -1 \pmod 4, \end{cases}$$

and

$$N_p(E_2) = 2^t \cdot 13 \text{ where } t = \begin{cases} 5 \text{ if } x \equiv 1 \pmod 4, \\ 6 \text{ if } x \equiv -1 \pmod 4. \end{cases}$$

There are 6 newforms at level $2^5 \cdot 13$ and 16 at level $2^6 \cdot 13$.

It is often possible to obtain bound for the exponent p. (Samir's notes, section 6). Let $E_1 \sim_p f_1$ and $E_2 \sim_p f_2$. Let c_l be the l-th coefficient of f_1 and d_l be the l-th coefficient of f_2. Define

$$B_l'(f_1) = \text{Norm}_{K/\mathbb{Q}}((l+1)^2 - c_l^2) \prod_{x,y \in \mathbb{F}_l} \text{Norm}_{K/\mathbb{Q}}(a_l(E_1) - c_l),$$

and

$$B_l(f_1) = \begin{cases} l \cdot B_l'(f_1) \text{ if } f \text{ is not rational,} \\ B_l'(f_1) \text{ if } f \text{ is rational.} \end{cases}$$

Similarly for f_2. We have $p \mid \gcd(B_l(f_1), B_l(f_2))$. The above argument implies that if there exists a solution of $x^2 + 13^m = y^p$ then $p \in \{3, 5\}$.

$$x^2 + 13^m = 2y^p$$

Here we have the following two Frey curves

$$\begin{aligned} E_1 : \quad & Y^2 = X^3 + 2xX^2 + 2y^pX, \\ E_2 : \quad & Y^2 = X^3 + 2xX^2 + (x^2 - 2y^p)X, \end{aligned}$$

and $N_p(E_1) = N_p(E_2) = 2^7 \cdot 13$. There are 28 newforms at level $2^7 \cdot 13$. The previous argument does not provide bound for the exponent p in this case. There are only a few pairs of newforms for which it happens.

$$x^2 + 13^m = 2y^p$$

Here we have the following two Frey curves

$$E_1: \quad Y^2 = X^3 + 2xX^2 + 2y^pX,$$
$$E_2: \quad Y^2 = X^3 + 2xX^2 + (x^2 - 2y^p)X,$$

and $N_p(E_1) = N_p(E_2) = 2^7 \cdot 13$. There are 28 newforms at level $2^7 \cdot 13$. The previous argument does not provide bound for the exponent p in this case. There are only a few pairs of newforms for which it happens. Rewrite the Frey curves as follows

$$E_1: \quad Y^2 = X^3 + 2xX^2 + (x^2 + 13^m)X,$$
$$E_2: \quad Y^2 = X^3 + 2xX^2 + (-13^m)X.$$

We get congruence conditions for m. We have $\mathrm{ord}_7(13) = 2$ and $\mathrm{ord}_{11}(13) = 10$.

pair	$l=7$	$l=11$
$(4,7)$	$0 \pmod 2$	
$(5,19)$	$0 \pmod 2$	
$(7,4)$	$0 \pmod 2$	
$(8,12)$		$0,2,4,6,8 \pmod{10}$
$(11,18)$		$0,2,4,6,8 \pmod{10}$
$(12,8)$		$0,2,4,6,8 \pmod{10}$
$(18,11)$		$0,2,4,6,8 \pmod{10}$
$(19,5)$	$0 \pmod 2$	

Therefore $m \equiv 0 \pmod 2$.

$$x^2 + 13^{2k} = 2y^p$$

The equation $x^2 + q^{2k} = 2y^p$.

$$\delta_4 = \begin{cases} 1 \text{ if } p \equiv 1 \pmod 4, \\ -1 \text{ if } p \equiv 3 \pmod 4. \end{cases}$$

$$\delta_8 = \begin{cases} 1 \text{ if } p \equiv 1 \text{ or } 3 \pmod 8, \\ -1 \text{ if } p \equiv 5 \text{ or } 7 \pmod 8. \end{cases}$$

$$y = u^2 + v^2,$$
$$x = \Re((1+i)(u+iv)^p) =: F_p(u,v),$$
$$q^k = \Im((1+i)(u+iv)^p) =: G_p(u,v).$$

Reducibility

$$\begin{aligned}(u-\delta_4 v) &\mid F_p(u,v),\\ (u+\delta_4 v) &\mid G_p(u,v).\end{aligned}$$

$$(u-\delta_4 v) \mid F_p(u,v),$$
$$(u+\delta_4 v) \mid G_p(u,v).$$

Example: $p=5$.

$$F_5(u,v) = (u-v)(u^4-4u^3v-14u^2v^2-4uv^3+v^4),$$
$$G_5(u,v) = (u+v)(u^4+4u^3v-14u^2v^2+4uv^3+v^4).$$

There exists $s \in \{0, 1, \ldots, k\}$ such that

$$\begin{aligned} u + \delta_4 v &= q^s, \\ H_p(u, v) &= q^{k-s}, \end{aligned} \qquad (1)$$

or

$$\begin{aligned} u + \delta_4 v &= -q^s, \\ H_p(u, v) &= -q^{k-s}, \end{aligned} \qquad (2)$$

where $H_p(u, v) = \frac{G_p(u,v)}{u+\delta_4 v}$.

We have $\deg H_p(\pm q^s - \delta_4 v, v) = p - 1$ and

$$H_p(\pm q^s - \delta_4 v, v) = \pm\delta_8 2^{\frac{p-1}{2}} p v^{p-1} + q^s p \widehat{H}_p(v) + q^{s(p-1)},$$

where $\widehat{H}_p \in \mathbb{Z}[X]$.

We have $\deg H_p(\pm q^s - \delta_4 v, v) = p - 1$ and

$$H_p(\pm q^s - \delta_4 v, v) = \pm\delta_8 2^{\frac{p-1}{2}} p v^{p-1} + q^s p \widehat{H}_p(v) + q^{s(p-1)},$$

where $\widehat{H}_p \in \mathbb{Z}[X]$.

Equations (1) and (2) imply

$$\pm\delta_8 2^{\frac{p-1}{2}} p v^{p-1} + q^s p \widehat{H}_p(v) + q^{s(p-1)} = \pm q^{k-s}.$$

We have $\deg H_p(\pm q^s - \delta_4 v, v) = p-1$ and

$$H_p(\pm q^s - \delta_4 v, v) = \pm\delta_8 2^{\frac{p-1}{2}} p v^{p-1} + q^s p \widehat{H}_p(v) + q^{s(p-1)},$$

where $\widehat{H}_p \in \mathbb{Z}[X]$.
Equations (1) and (2) imply

$$\pm\delta_8 2^{\frac{p-1}{2}} p v^{p-1} + q^s p \widehat{H}_p(v) + q^{s(p-1)} = \pm q^{k-s}.$$

The following cases are possible

We have $\deg H_p(\pm q^s - \delta_4 v, v) = p - 1$ and

$$H_p(\pm q^s - \delta_4 v, v) = \pm\delta_8 2^{\frac{p-1}{2}} p v^{p-1} + q^s p \widehat{H}_p(v) + q^{s(p-1)},$$

where $\widehat{H}_p \in \mathbb{Z}[X]$.
Equations (1) and (2) imply

$$\pm\delta_8 2^{\frac{p-1}{2}} p v^{p-1} + q^s p \widehat{H}_p(v) + q^{s(p-1)} = \pm q^{k-s}.$$

The following cases are possible

- $p = q, s = k - 1,$

We have $\deg H_p(\pm q^s - \delta_4 v, v) = p - 1$ and

$$H_p(\pm q^s - \delta_4 v, v) = \pm \delta_8 2^{\frac{p-1}{2}} p v^{p-1} + q^s p \widehat{H}_p(v) + q^{s(p-1)},$$

where $\widehat{H}_p \in \mathbb{Z}[X]$.
Equations (1) and (2) imply

$$\pm \delta_8 2^{\frac{p-1}{2}} p v^{p-1} + q^s p \widehat{H}_p(v) + q^{s(p-1)} = \pm q^{k-s}.$$

The following cases are possible

- $p = q, s = k - 1,$
- $p \neq q, s = 0$ or $s = k.$

All solutions of the equation $x^2 + q^{2k} = 2y^p$ with $3 \leq q^k \leq 501$ are as follows

$$(x, y, q, k, p) \in \{(3, 5, 79, 1, 5), (9, 5, 13, 1, 3), (13, 5, 3, 2, 3), (55, 13, 37, 1, 3), (79, 5, 3, 1, 5), (99, 17, 5, 1, 3), (161, 25, 73, 1, 3), (249, 5, 307, 1, 7), (351, 41, 11, 2, 3), (545, 53, 3, 3, 3), (649, 61, 181, 1, 3), (1665, 113, 337, 1, 3), (2431, 145, 433, 1, 3), (5291, 241, 19, 1, 3), (275561, 3361, 71, 1, 3)\}.$$

It remains to deal with

$$x^2 + 13^{2k} = 2y^p,$$

with $k \geq 3$.

Linear forms in two logs

Theorem. *If* $x^2 + 13^{2k} = 2y^p$ *admits a relatively prime solution* $(x, y) \in \mathbb{N}^2$ *then we have* $p \leq 3203$ *if* $u + \delta_4 v = \pm 13^k, k \geq 3$.

We get

$$\frac{13^k}{2} \leq \frac{|u| + |v|}{2} \leq \sqrt{\frac{u^2 + v^2}{2}} = \sqrt{\frac{y}{2}}.$$

We have

$$\left| \frac{x + 13^k i}{x - 13^k i} - 1 \right| = \frac{2 \cdot 13^k}{\sqrt{x^2 + 13^{2k}}} \leq \frac{2\sqrt{y}}{y^{p/2}} = \frac{2}{y^{\frac{p-1}{2}}},$$

and

$$\frac{x + 13^k i}{x - 13^k i} = \frac{(1+i)(u+iv)^p}{(1-i)(u-iv)^p} = i \left(\frac{u+iv}{u-iv} \right)^p.$$

Finally

$$\left| i \left(\frac{u+iv}{u-iv} \right)^p - 1 \right| \geq \frac{1}{2} \left| \log i \left(\frac{u+iv}{u-iv} \right)^p \right|.$$

Lemma. *In case of* $p > 3$ *there is no solution of* (1) *and* (2) *with* $s = 0$.

Lemma. *In case of $p > 3$ there is no solution of* (1) *and* (2) *with $s = 0$.*

Proof. In case of (1) if $s = 0$, then $u = 1 - \delta_4 v$. Observe that by the definition of H_p

- if $v \equiv 0 \pmod{13}$, then $H_p(1-\delta_4 v, v) \equiv 1 \pmod{13}$,
- if $v \equiv 1 \pmod{13}$ and $p \equiv 1 \pmod{4}$, then $H_p(1-\delta_4 v, v) \equiv 1 \pmod{13}$,
- if $v \equiv 1 \pmod{13}$ and $p \equiv 3 \pmod{4}$, then $H_p(1-\delta_4 v, v) \equiv \pm 5 \pmod{13}$,
- if $v \equiv 2 \pmod{13}$ and $p \equiv 1 \pmod{4}$, then $H_p(1-\delta_4 v, v) \equiv \pm 1 \pmod{13}$,
- if $v \equiv 2 \pmod{13}$ and $p \equiv 3 \pmod{4}$, then $H_p(1-\delta_4 v, v) \equiv 7, 8 \pmod{13}$.
- if $v \equiv 3 \pmod{13}$ and $p \equiv 1 \pmod{4}$, then $H_p(1-\delta_4 v, v) \equiv 1, 9 \pmod{13}$,
- if $v \equiv 3 \pmod{13}$ and $p \equiv 3 \pmod{4}$, then $H_p(1-\delta_4 v, v) \equiv 6, 12 \pmod{13}$.
- if $v \equiv 4 \pmod{13}$ and $p \equiv 1 \pmod{4}$, then $H_p(1-\delta_4 v, v) \equiv 1, 7 \pmod{13}$,
- if $v \equiv 4 \pmod{13}$ and $p \equiv 3 \pmod{4}$, then $H_p(1-\delta_4 v, v) \equiv 7, 8 \pmod{13}$.
- etc.

Thus if $p > 3$ then $H_p(1-\delta_4 v, v) \not\equiv 0 \pmod{13}$. We remark that $u + \delta_4 v = -13^k$ is not possible because $-1 \equiv H_p(-13^k - \delta_4 v, v) \equiv 13^{k(p-1)} \equiv 1 \pmod{p}$. □

Remaining possibility

$$u + \delta_4 v = 13^k,$$
$$H_p(u, v) = 1,$$
$$x = F_p(13^k - \delta_4 v, v).$$

Corresponding Frey curves

$$E_1: \quad Y^2 = X^3 + 2F_p(13^k - \delta_4 v)X^2 + (F_p(13^k - \delta_4 v)^2 + 13^{2k})X,$$
$$E_2: \quad Y^2 = X^3 + 2F_p(13^k - \delta_4 v)X^2 + (-13^{2k})X.$$

Back to Frey curves

Remaining possibility

$$\begin{aligned} u+\delta_4 v &= 13^k, \\ H_p(u,v) &= 1, \\ x &= F_p(13^k-\delta_4 v, v). \end{aligned}$$

Corresponding Frey curves

$$\begin{aligned} E_1: \quad & Y^2 = X^3 + 2F_p(13^k-\delta_4 v)X^2 + (F_p(13^k-\delta_4 v)^2 + 13^{2k})X, \\ E_2: \quad & Y^2 = X^3 + 2F_p(13^k-\delta_4 v)X^2 + (-13^{2k})X. \end{aligned}$$

”Good” primes: primes of the form $l = np+1$ or primes l for which $\mathrm{ord}_l(13)$ is ”small”. Using such primes and the method of Kraus we can exclude all primes $p \in \{7, \dots, 3203\}$.

Product of terms in AP

$$n(n+d)\cdots(n+(k-1)d) = by^2$$

where $\gcd(n,d) = 1$ and $P(b) \leq k$.
We have

$$n + id = a_i x_i^2 \text{ for } 0 \leq i < k.$$

$$n(n+d)\cdots(n+(k-1)d) = by^2$$

where $\gcd(n,d) = 1$ and $P(b) \leq k$.
We have

$$n + id = a_i x_i^2 \text{ for } 0 \leq i < k.$$

Theorem (Hirata-Kohno, Laishram, Shorey,Tijdeman)**.** *The above equation with $d > 1$, $P(b) = k$ and $7 \leq k \leq 100$ implies that $(a_0, a_1, \ldots, a_{k-1})$ is among the following tuples or their mirror images.*

$$\begin{aligned}
k = 7: &\quad (2,3,1,5,6,7,2), (3,1,5,6,7,2,1), (1,5,6,7,2,1,10),\\
k = 13: &\quad (3,1,5,6,7,2,1,10,11,3,13,14,15),\\
&\quad (1,5,6,7,2,1,10,11,3,13,14,15,1),\\
k = 19: &\quad (1,5,6,7,2,1,10,11,3,13,14,15,1,17,2,19,5,21,22),\\
k = 23: &\quad (5,6,7,2,1,10,11,3,13,14,15,1,17,2,19,5,21,22,23,6,1,26,3),\\
&\quad (6,7,2,1,10,11,3,13,14,15,1,17,2,19,5,21,22,23,6,1,26,3,7).
\end{aligned}$$

The cases $k = 5, P(b) = 5$ and
$k = 7, (2, 3, 1, 5, 6, 7, 2), (3, 1, 5, 6, 7, 2, 1), (1, 5, 6, 7, 2, 1, 10)$.

Theorem (Bennett). *If n and d are coprime nonzero integers, then the Diophantine equation*

$$n(n + d)(n + 2d)(n + 3d)(n + 4d) = by^l$$

has no solutions in nonzero integers b, y and l with $l \geq 2$ and $P(b) \leq 3$.

$$T = \{(a_0, a_1, a_2, a_3, a_4)|a_i = 2^\alpha 3^\beta 5^\gamma\}.$$

WLOG $5|a_1$ or $5|a_2$.

Eliminate tuples

- $(6, -5, 1, 3, 2)$.
- Congruence arguments.
- Rank 0 elliptic curves.

- $(6, -5, 1, 3, 2)$.
- Congruence arguments.
- Rank 0 elliptic curves.

The only possible tuples are

$$(2,5,2,-1,-1), (2,5,-3,-1,-1), (3,5,-2,-1,-1), (6,5,1,3,2).$$

Using $n + 2d = 2x_2^2$ and $n + 3d = -x_3^2$ we obtain

$$\begin{aligned} x_3^2 + 3x_2^2 &= x_0^2, \\ x_3^2 + 4x_2^2 &= 5x_1^2, \\ 2x_3^2 + 2x_2^2 &= x_4^2. \end{aligned}$$

Remark: Rank$(J) = 2$.

After factorization we get

$$(x_3 + ix_2)(x_3 + 2ix_2)(x_3^2 + 3x_2^2) = \delta\Box,$$

where $\delta \in \{-3 \pm i, -1 \pm 3i, 1 \pm 3i, 3 \pm i\}$.

After factorization we get

$$(x_3 + ix_2)(x_3 + 2ix_2)(x_3^2 + 3x_2^2) = \delta\Box,$$

where $\delta \in \{-3 \pm i, -1 \pm 3i, 1 \pm 3i, 3 \pm i\}$.

Elliptic Chabauty's method: implemented in MAGMA by Nils Bruin.

(lecture by Nils Bruin)

After factorization we get

$$(x_3 + ix_2)(x_3 + 2ix_2)(x_3^2 + 3x_2^2) = \delta\square,$$

where $\delta \in \{-3 \pm i, -1 \pm 3i, 1 \pm 3i, 3 \pm i\}$.
Elliptic Chabauty's method: implemented in MAGMA by Nils Bruin.
(lecture by Nils Bruin)

- $-3 \pm i, 3 \pm i$: RankBound $= 0$.

Elliptic curves

After factorization we get

$$(x_3 + ix_2)(x_3 + 2ix_2)(x_3^2 + 3x_2^2) = \delta\Box,$$

where $\delta \in \{-3 \pm i, -1 \pm 3i, 1 \pm 3i, 3 \pm i\}$.
Elliptic Chabauty's method: implemented in MAGMA by Nils Bruin.
(lecture by Nils Bruin)

- $-3 \pm i, 3 \pm i$: RankBound $= 0$.
- $-1 - 3i$: RankBound $= 1$. Using $p = 13$ we obtain that the only solution with $x_3/x_2 \in \mathbb{Q}$ is -1.

Elliptic curves

After factorization we get

$$(x_3 + ix_2)(x_3 + 2ix_2)(x_3^2 + 3x_2^2) = \delta\Box,$$

where $\delta \in \{-3 \pm i, -1 \pm 3i, 1 \pm 3i, 3 \pm i\}$.
Elliptic Chabauty's method: implemented in MAGMA by Nils Bruin.
(lecture by Nils Bruin)

- $-3 \pm i, 3 \pm i$: RankBound $= 0$.
- $-1 - 3i$: RankBound $= 1$. Using $p = 13$ we obtain that the only solution with $x_3/x_2 \in \mathbb{Q}$ is -1.
- $-1 + 3i$: RankBound $= 1$. Using again $p = 13$ it follows that $x_3/x_2 = 1$.

After factorization we get

$$(x_3 + ix_2)(x_3 + 2ix_2)(x_3^2 + 3x_2^2) = \delta\Box,$$

where $\delta \in \{-3 \pm i, -1 \pm 3i, 1 \pm 3i, 3 \pm i\}$.
Elliptic Chabauty's method: implemented in MAGMA by Nils Bruin. (lecture by Nils Bruin)

- $-3 \pm i, 3 \pm i$: RankBound $= 0$.
- $-1 - 3i$: RankBound $= 1$. Using $p = 13$ we obtain that the only solution with $x_3/x_2 \in \mathbb{Q}$ is -1.
- $-1 + 3i$: RankBound $= 1$. Using again $p = 13$ it follows that $x_3/x_2 = 1$.
- $1 - 3i$: RankBound $= 1$. Here we have $x_3/x_2 = 1$.

After factorization we get

$$(x_3 + ix_2)(x_3 + 2ix_2)(x_3^2 + 3x_2^2) = \delta\square,$$

where $\delta \in \{-3 \pm i, -1 \pm 3i, 1 \pm 3i, 3 \pm i\}$.
Elliptic Chabauty's method: implemented in MAGMA by Nils Bruin. (lecture by Nils Bruin)

- $-3 \pm i, 3 \pm i$: RankBound $= 0$.
- $-1 - 3i$: RankBound $= 1$. Using $p = 13$ we obtain that the only solution with $x_3/x_2 \in \mathbb{Q}$ is -1.
- $-1 + 3i$: RankBound $= 1$. Using again $p = 13$ it follows that $x_3/x_2 = 1$.
- $1 - 3i$: RankBound $= 1$. Here we have $x_3/x_2 = 1$.
- $1 + 3i$: RankBound $= 1$. In this case $x_3/x_2 = -1$.

After factorization we get

$$(x_3 + ix_2)(x_3 + 2ix_2)(x_3^2 + 3x_2^2) = \delta\square,$$

where $\delta \in \{-3 \pm i, -1 \pm 3i, 1 \pm 3i, 3 \pm i\}$.
Elliptic Chabauty's method: implemented in MAGMA by Nils Bruin.
(lecture by Nils Bruin)

- $-3 \pm i, 3 \pm i$: RankBound $= 0$.
- $-1 - 3i$: RankBound $= 1$. Using $p = 13$ we obtain that the only solution with $x_3/x_2 \in \mathbb{Q}$ is -1.
- $-1 + 3i$: RankBound $= 1$. Using again $p = 13$ it follows that $x_3/x_2 = 1$.
- $1 - 3i$: RankBound $= 1$. Here we have $x_3/x_2 = 1$.
- $1 + 3i$: RankBound $= 1$. In this case $x_3/x_2 = -1$.

The AP is $[8, 5, 2, -1, -4]$, that is $n = 8$ and $d = -3$.

Magma computation

$$P < x > := PolynomialRing(Rationals());$$

$$N < i > := NumberField(x^2 + 1);$$

$$R < X > := PolynomialRing(N);$$

$$P1 := ProjectiveSpace(Rationals(), 1);$$

$$C := HyperellipticCurve((1 + 3 * i) * (X + i) * (X + 2 * i) * (X^2 + 3));$$

$$E, toE := EllipticCurve(C);$$

$$Em, EtoEm := MinimalModel(E);$$

$P < x > := PolynomialRing(Rationals());$

$N < i > := NumberField(x^2 + 1);$

$R < X > := PolynomialRing(N);$

$P1 := ProjectiveSpace(Rationals(), 1);$

$C := HyperellipticCurve((1 + 3 * i) * (X + i) * (X + 2 * i) * (X^2 + 3));$

$E, toE := EllipticCurve(C);$

$Em, EtoEm := MinimalModel(E);$

$\mathbf{y^2 = x^3 + i * x^2 + (5 * i - 7) * x + (4 * i - 6)}$

$umap := map < C -> P1|[C.1, C.3] >;$

$U := Expand(Inverse(toE * EtoEm) * umap);$

$success, G, mwmap := PseudoMordellWeilGroup(Em);$

$NC, VC, RC, CC := Chabauty(mwmap, U, 13);$

Magma computation

$P < x >:= PolynomialRing(Rationals());$

$N < i >:= NumberField(x^2 + 1);$

$R < X >:= PolynomialRing(N);$

$P1 := ProjectiveSpace(Rationals(), 1);$

$C := HyperellipticCurve((1 + 3 * i) * (X + i) * (X + 2 * i) * (X^2 + 3));$

$E, toE := EllipticCurve(C);$

$Em, EtoEm := MinimalModel(E);$

$\mathbf{y^2 = x^3 + i * x^2 + (5 * i - 7) * x + (4 * i - 6)}$

$umap := map < C- > P1|[C.1, C.3] >;$

$U := Expand(Inverse(toE * EtoEm) * umap);$

$success, G, mwmap := PseudoMordellWeilGroup(Em);$

$NC, VC, RC, CC := Chabauty(mwmap, U, 13);$

$\mathbf{NC = 2, VC = \{G.1 \pm G.2\}, RC = 2}$

$forall\{pr : pr\ in\ PrimeDivisors(RC)|IsPSaturated(mwmap, pr)\};$

$\{EvaluateByPowerSeries(U, mwmap(gp)) : gp\ in\ VC\};$

$\{(\mathbf{-1 : 1})\}$

Paper by Nils Bruin, Kálmán Győry, Lajos Hajdu, Szabolcs Tengely (2006).

Theorem. *Let $k \geq 4$ and $L \geq 2$. There are only finitely many k-term integral arithmetic progressions $(h_0, h_1, \ldots, h_{k-1})$ such that $\gcd(h_0, h_1) = 1$ and $h_i = x_i^{l_i}$ with some $x_i \in \mathbb{Z}$ and $2 \leq l_i \leq L$ for $i = 0, 1, \ldots, k-1$.*

In case of $(l_0, l_1, l_2, l_3) = (2, 2, 2, 3)$

$$((u^2 - 2uv - v^2)f(u,v))^2, ((u^2 + v^2)f(u,v))^2, ((u^2 + 2uv - v^2)f(u,v))^2, (f(u,v))^3$$

is an arithmetic progression for any $u, v \in \mathbb{Z}$, where $f(u,v) = u^4 + 8u^3v + 2u^2v^2 - 8uv^3 + v^4$.

Let $x_0^3, x_1^2, x_2^3, x_3^2$ be consecutive terms of an arithmetic progression with $\gcd(x_0, x_1, x_2, x_3) = 1$. We have

$$x_1^2 = \frac{x_0^3 + x_2^3}{2},$$
$$x_3^2 = \frac{-x_0^3 + 3x_2^3}{2}.$$

Theorem. *Let $\mathcal{C}$ be the curve given by*

$$Y^2 = -X^6 + 2X^3 + 3.$$

Then $\mathcal{C}(\mathbb{Q}) = \{(-1, 0), (1, \pm 2)\}$.

Solutions are given by
$(x_0, x_1, x_2, x_3) \in \{(-2t^2, 0, 2t^2, \pm 4t^3), (t^2, \pm t^3, t^2, \pm t^3)\}$ for some $t \in \mathbb{Z}$.