Combinatorial numbers in recurrence sequences

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Genealogy

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$$\text{Turán} \xrightarrow{\text{PhD supervisor}} \text{Győry} \dashrightarrow^{\text{MSc supervisor}} \text{Tengely}$$

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$$\begin{tikzcd} \text{Turán} \arrow[r, "\text{PhD supervisor}"] & \text{Győry} \arrow[r, dashed, "\text{MSc supervisor}"] & \text{Tengely} \end{tikzcd}$$

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"Almost scientific grandchild of Turán"

Integer sequences

Problem: find intersection of integer sequences
Some well-known sequences:

- perfect powers
- binomial coefficients
- Fibonacci sequence, recurrence sequences

We will consider the equation

$$L_n = \binom{x}{5}$$

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Related results

- Cohn and independently Wyler: the only squares in the Fibonacci sequence are $F_0 = 0, F_1 = F_2 = 1, F_{12} = 144$.
- Alfred and independently Cohn: perfect squares in the Lucas sequence.
- Cohn and independently Pethő: perfect squares in the Pell sequence.
- London and Finkelstein and independently Pethő: the only cubes in the Fibonacci sequence are $F_0 = 0, F_1 = F_2 = 1$ and $F_6 = 8$.
- Bugeaud, Mignotte and Siksek: combination of Baker's method, modular approach and some classical techniques to show that the perfect powers in the Fibonacci sequence are 0,1,8 and 144, and the perfect powers in the Lucas sequence are 1 and 4.
- Szalay: solved the equations $F_n, L_n, P_n = \binom{x}{3}$ and $F_n, L_n = \binom{x}{4}$.
- Kovács: solved the equations $P_n = \binom{x}{4}$ and $F_n = \Pi_4(x)$.

Similar combinatorial Diophantine problems

Many results, we mention only a few mathematicians working on this subject: Bennett, Bilu, Bremner, Bugeaud, Győry, Hajdu, Hanrot, Kovács, Luca, Mignotte, Olajos, Pethő, Pintér, Rakaczki, Saradha, Shorey, Siksek, Stewart, Stoll, Stroeker, Szalay, Tijdeman, Tzanakis, De Weger.

Main result

We consider the Diophantine equation

$$L_n = \binom{x}{5}. \tag{1}$$

Theorem

The only positive solution of equation (1) is $(n, x) = (1, 5)$.

We will use the following well known property of the sequences F_n and L_n :

$$L_n^2 - 5F_n^2 = 4(-1)^n.$$

We have that

$$\binom{x}{5}^2 \pm 4 = 5F_n^2.$$

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The above equation can be reduced to two genus two curves as follows

$$\mathcal{C}^+ : Y^2 = X^2(X+15)^2(X+20) + 180000000 \tag{2}$$

and

$$\mathcal{C}^- : Y^2 = X^2(X+15)^2(X+20) - 180000000, \tag{3}$$

where $Y = 5^3 5! F_n$ and $X = 5x^2 - 20x$.

Theorem

(a) The integral solutions of equation (2) are

$$(X, Y) \in \{(25, -15000), (25, 15000)\}.$$

(b) There are no integral solution of equation (3).

To prove the above results we will follow the paper by Bugeaud, Mignotte, Siksek, Stoll and Tengely. They combined Baker's method and the so-called Mordell-Weil sieve to solve

$$\binom{x}{2} = \binom{y}{5}$$

and

$$x^2 - x = y^5 - y.$$

Proof of part (a)

Using MAGMA (procedures based on Stoll's papers) we obtain that $J(\mathbb{Q})^+$ is free of rank 1 with Mordell-Weil basis given by

$$D = (25, 15000) - \infty.$$

Classical Chabauty's method can be applied.

$$\mathcal{C}^+(\mathbb{Q}) = \{\infty, (25, \pm 15000)\}.$$

Proof of part (b)

Using MAGMA we determine a Mordell-Weil basis which is given by

$$D_1 = (\omega_1, -200\omega_1) + (\overline{\omega}_1, -200\overline{\omega}_1) - 2\infty,$$
$$D_2 = (\omega_2, 120000) + (\overline{\omega}_2, 120000) - 2\infty,$$

where ω_1 is a root of the polynomial $x^2 - 5x + 1500$ and ω_2 is a root of $x^2 + 195x + 13500$.
Let $f = x^2(x+15)^2(x+20) - 180000000$ and α be a root of f.
We have

$$x - \alpha = \kappa\xi^2,$$

such that $\kappa \in \{1, \alpha^2 - 5\alpha + 1500, \alpha^2 + 195\alpha + 13500, \alpha^4 + 190\alpha^3 + 14025\alpha^2 + 225000\alpha + 20250000\}$.

By local arguments it is possible to restrict the set. In our case one can eliminate

$$\alpha^2 - 5\alpha + 1500, \quad \alpha^2 + 195\alpha + 13500$$

by local computations in $\mathbb{Q}_2$ and

$$\alpha^4 + 190\alpha^3 + 14025\alpha^2 + 225000\alpha + 20250000$$

by local computations in $\mathbb{Q}_3$. It remains to deal with the case $\kappa = 1$. By Baker's method we get a large upper bound for $\log |x|$:

$$1.58037 \times 10^{285}.$$

The set of known rational points on the curve (3) is $\{\infty\}$. Let W be the image of this set in $J(\mathbb{Q})^-$. Applying the Mordell-Weil sieve implemented by Bruin and Stoll we obtain that

$$\jmath(C(\mathbb{Q})) \subseteq W + BJ(\mathbb{Q})^-,$$

where

$$B = 2^6 \cdot 3^2 \cdot 5^2 \cdot 7^2 \cdot 11^2 \cdot 13^2 \cdot 19 \cdot 23 \cdot 31 \cdot 41 \cdot 43 \cdot 47 \cdot 61 \cdot 67 \cdot 79 \cdot 83 \cdot 109 \cdot 113 \cdot 127,$$

that is

$$B = 678957252681082328769065398948800.$$

Now we use an extension of the Mordell-Weil sieve due to Samir Siksek to obtain a very long decreasing sequence of lattices in $\mathbb{Z}^2$ to obtain a lower bound for possible unknown rational points.

If (x, y) is an unknown integral point, then

$$\log|x| \geq 7.38833 \times 10^{1076}.$$

This contradicts the bound for $\log|x|$ we obtained by Baker's method.

Proof of the main theorem

We have that $X = 25$ and we also have that $X = 5x^2 - 20x$. We obtain that $x \in \{-1, 5\}$.

$$1 = L_1 = \binom{5}{5}.$$

Balancing numbers

A positive integer n is called a balancing number if $1 + 2 + \ldots + (n-1) = (n+1) + (n+2) + \ldots + (n+k)$ for some $k \in \mathbb{N}$. The sequence of balancing numbers is denoted by B_m for $m \in \mathbb{N}$.

Theorem

The Diophantine equation

$$B_m = x(x+1)(x+2)(x+3)(x+4) \quad m \geq 0, x \in \mathbb{Z}$$

has no solution.

Related results

- Behera and Panda proved many interesting results,
- Liptai proved that there are neither Fibonacci nor Lucas balancing numbers,
- Bérczes, Liptai and Pink: further generalization in this direction,
- Kovács, Liptai and Olajos proved some general finiteness results,
- Liptai, Luca, Pintér and Szalay introduced the concept of (k, l)-power numerical center and obtained certain effective and ineffective finiteness results

Liptai proved that the integers B_m satisfy the following equation

$$z^2 - 8y^2 = 1$$

for some integer z. So one has to determine all solution of the equation

$$z^2 = 8(x(x+1)(x+2)(x+3)(x+4))^2 + 1.$$

Rewrite the latter equation as follows

$$z^2 = 8(x^2+4x)^2(x^2+4x+3)^2(x^2+4x+4) + 1.$$

Let $X = 2x^2 + 8x$.

We obtain that

$$\mathcal{C}: \quad Y^2 = X^2(X+6)^2(X+8)+4, \tag{4}$$

where $Y = 2z$. It remains to find all integral points on $\mathcal{C}$. The rank of the Jacobian of $\mathcal{C}$ is 2.

Lemma

The only integral solutions to the above equation are

$$(0, \pm 2), (-6, \pm 2), (-8, \pm 2).$$