Diophantine problems related to recurrence sequences

Szabolcs Tengely

Number Theory and its Applications
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Dedicated to
Kálmán Győry, Attila Pethő,
János Pintz, András Sárközy

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Research problem

$$F_n = \binom{x}{5},$$

where F_n is the Fibonacci sequence: $0, 1, 1, 2, 3, 5, \ldots$

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I would like to thank Prof. Michael Stoll for organising the workshop on "Rational Points 3" and for the discussions related to the present topic during the workshop on the arithmetic of genus 2 curves and Mordell-Weil sieve.

Some related results

Kálmán Győry: On the Diophantine equations $\binom{n}{2} = a^l$ and $\binom{n}{3} = a^l$, Mat. Lapok **14** (1963) 322-329.

It was proved that for $n > 2$ and $l > 1$ the equation

$$\binom{n}{2} = a^{2l}$$

has no solution. Also some results on the unsolvability of the equation

$$\binom{n}{3} = a^p$$

were discussed, where p is an odd prime.

Kálmán Győry: On the Diophantine equation $\binom{n}{k} = x^l$,
Acta Arith. **80** (1997) 289-295.

In 1951, Erdős proved that the equation $\binom{n}{k} = x^l$ has no integral solution with $x > 1, l > 1, k \geq 4$ and $n \geq 2k$.
Kálmán Győry resolved the remaining cases with $k = 2, 3$. He showed that the only solution is (where $(k, l) \neq (2, 2)$)

$$\binom{50}{3} = 140^2.$$

Attila Pethő: The Pell sequence contains only trivial perfect powers, Sets, graphs and numbers (Budapest, 1991), 561568.
Colloq. Math. Soc. János Bolyai, 60, North-Holland, Amsterdam, 1992.

By extending a result of Ljunggren, Attila Pethő proved that the only nontrivial solution of the equation

$$R_n = x^q$$

in integers n, x, q with $|x| > 1$ and $q \geq 2$ is

$$(n, x, q) = (7, 13, 2).$$

Here R_n is defined by $R_0 = 0, R_1 = 1$ and $R_{n+2} = 2R_{n+1} + R_n$ for $n \geq 0$.

Clemens Fuchs, Attila Pethő and Robert Tichy: On the Diophantine equation $G_n(x) = G_m(P(x))$: higher-order recurrences, Trans. Amer. Math. Soc. **355** (2003) 4657-4681.

$$(G_n(x))_{n=0}^{\infty}$$

degree d linear recurring sequence defined by

$$G_{n+d}(x) = A_{d-1}(x)G_{n+d-1}(x) + \ldots + A_0(x)G_n(x), \qquad \text{for } n \geq 0.$$

General finiteness conditions depending only on $G_0, \ldots, G_{d-1} \in \mathbb{K}[x]$, on $P \in \mathbb{K}[x]$ and on $A_0, \ldots, A_{d-1} \in \mathbb{K}[x]$ related to the equation

$$G_n(x) = G_m(P(x)).$$

Enrico Bombieri, Andrew Granville and János Pintz: Squares in arithmetic progressions
Duke Math. J. **66** (1992) 369-385.

$Q(N; q, a)$: number of squares in the arithmetic progression $qn + a \quad (n = 1, 2, \ldots, N)$

$$Q(N) = \max_{a,q \geq 1} Q(N; q, a).$$

It was conjectured by Erdős and proved by Szemerédi, that $Q(N) = o(N)$.
Enrico Bombieri, Andrew Granville and János Pintz proved that

$$Q(N) = O(N^{2/3}(\log N)^A).$$

András Sárközy: On multiplicative arithmetic functions satisfying a linear recursion
Studia Sci. Math. Hungar. **13** (1978) 79104.

Multiplicative arithmetic function: $f(mn) = f(m)f(n), \gcd(m, n) = 1$.
Determine all multiplicative arithmetic function which satisfy a linear recursion of finite order:

$$a_0 f(n) + a_1 f(n+1) + \ldots + a_k f(n+k) = 0, \quad n = 1, 2, \ldots$$

András Sárközy showed that $f(n)/n^h$ is periodic for some integer $h \geq 0$.
Moreover if $f(n) \not\equiv 0$ and $f(n) = o(n)$, then $f(n)$ is a character modulo m.

$U_n = AU_{n-1} + BU_{n-2}$: binary recurrence sequence.
Determine n, x such that

$$U_n = P(x),$$

where P is a polynomial.
Many results are in the literature, finiteness results, complete resolution in special cases
(Alfred, Bilu, Bugeaud, Cohn, Fuchs, Győry, Hajdu, Kovács, Luca, McDaniel, Mignotte, Ming, Nemes, Pethő, Pintér, Rakaczki, Schinzel, Shorey, Siksek, Stewart, Szalay, Tichy, Tijdeman)

2007, workshop in Leiden, Evertse and Tijdeman composed a list of open problems posed by the participants. Florian Luca posed the following problem:
show that the equation $F_n = \binom{m}{k}$ has only finitely many integer solutions (n, m, k) with $2 \leq k \leq m/2$.
Let $k = 5$.

$$F_n = \binom{x}{5},$$

where F_n is the Fibonacci sequence: $0, 1, 1, 2, 3, 5, \ldots$
By using the identity $L_n^2 - 5F_n^2 = 4(-1)^n$ we get

$$C^{+,-}: \quad y^2 = x^2(x+15)^2(x+20) \pm 36000000.$$

Following the paper by Bugeaud, Mignotte, Siksek, Stoll and Tengely: upper bound for the integral solutions by Baker's method (Matveev's bound)
lower bound for the integral solutions by Mordell-Weil sieve (implemented by Bruin and Stoll).

$$C^{-}: \quad y^2 = x^2(x+15)^2(x+20) - 36000000 = f(x).$$

Using MAGMA procedures based on Stoll's papers we obtain that $rank J(\mathbb{Q}) = 2$ and

$$\begin{aligned} D_1 &= (25, 3000) - \infty \\ D_2 &= \left(\frac{-5\sqrt{-15}-75}{2}, -1500\sqrt{-15}+1500\right) + \\ &\quad \left(\frac{5\sqrt{-15}-75}{2}, 1500\sqrt{-15}+1500\right) - 2\infty \end{aligned}$$

are generators of the Mordell-Weil group.

Representatives of $J(\mathbb{Q})/2J(\mathbb{Q})$ are $0, D_1, D_2, D_1 + D_2$, $f(\alpha) = 0$.
We have

$$x - \alpha = \kappa \square$$

By Baker's theory we have the following bounds:

repr.	κ	bound for $\log x$
0	1	9.28×10^{279}
D_1	$25 - \alpha$	4.53×10^{599}
D_2	$1500 + 75\alpha + \alpha^2$	eliminated in $\mathbb{Q}_2$
$D_1 + D_2$	$3900 - 81\alpha + \alpha^2$	eliminated in $\mathbb{Q}_3$

By Mordell-Weil sieve we obtain that

$$\jmath(C(\mathbb{Q})) \subseteq W + BJ(\mathbb{Q})$$

where W is the image of the known points $\{(25, \pm 3000), \infty\}$ and $B = 2^2 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 29$.

To obtain B we used the Mordell-Weil sieve with the following primes:

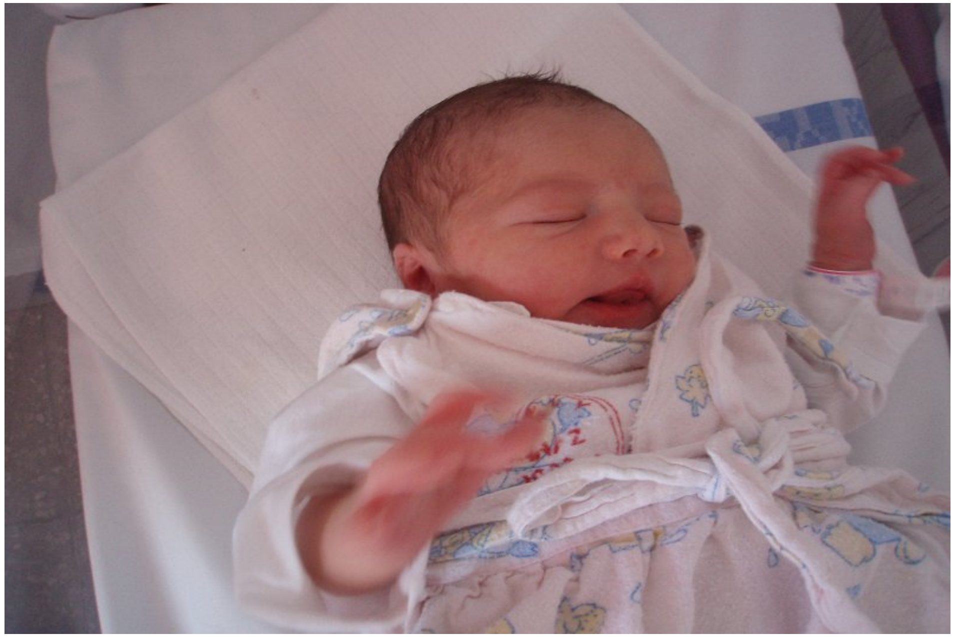
$$
\begin{aligned}
& 2^7, 5^2, 7, 11, 13, 17, 29, 47, 73 \\
, \quad & 101, 139, 151, 163, 179, 211, 257, 347, \\
& 509, 523, 617, 631, 769, 829, 877, 971, \\
& 1327, 1567, 1667, 1747, 1877, 2099, 2273, \\
& 2287, 2347, 2521, 2591, 2707, 2953, 3067, \\
& 3119, 3229, 3259, 4441, 4651, 4663, 5387, \\
& 6277, 6991, 9859, 10781, 10847, 11447, 12071, \\
& 14653, 14831, 15973, 17359, 19207.
\end{aligned}
$$

$$\phi : \mathbb{Z}^2 \to J(\mathbb{Q}) \quad \phi(a_1, a_2) = a_1 D_1 + a_2 D_2,$$

and $\jmath(C(\mathbb{Q})) \subseteq W + \phi(B\mathbb{Z}^2)$.

$$B\mathbb{Z}^2 = L_0 \supsetneq L_1 \supsetneq L_2 \ldots \supsetneq L_k$$

such that $\jmath(C(\mathbb{Q})) \subseteq W + \phi(L_j)$.
Lower bound obtained this way: 0.69×10^{617}.



Hanna Tengely
2nd October 2010
3.15 kg, 51 cm