On Composite Rational Functions

Szabolcs Tengely

Number Theory Seminar, Debrecen
(joint work with Attila Pethő)

23/03/2012

Research supported by OTKA PD75264 and János
Bolyai Research Scholarship of the Hungarian
Academy of Sciences

Introduction

We are interested in $f \in k(x)$ that are decomposable as rational functions, i.e. for which

$$f(x) = g(h(x))$$

with $g, h \in k(x)$, $\deg g, \deg h \geq 2$ holds.
Such a decomposition is only unique up to a linear fractional transformation

$$\lambda = \frac{ax + b}{cx + d}$$

with $ad - bc = \pm 1$, since we may always replace $g(x)$ by $g(\lambda(x))$ and $h(x)$ by $\lambda^{-1}(h(x))$ without affecting the equation $f(x) = g(h(x))$.

Related results

Schinzel conjectured that if for fixed g the polynomial $g(h(x))$ has at most l non-constant terms, then the number of terms of h is bounded only in terms of l.

A more general form of this conjecture was proved by **Zannier** in 2008. He proved that if one starts with a positive integer l, then one can describe effectively all decompositions of polynomials $f \in k[x]$ having at most l non-constant terms if one excludes the inner function h being of the exceptional shape $ax^n + b, a, b \in k, n \in \mathbb{N}$. This description was "algorithmic".

In this talk we are interested in rational functions

$$f = \frac{P}{Q}$$

with a **bounded number of zeros and poles** (i.e. the number of distinct roots of P, Q in a reduced expression of f is bounded).

We assume that the number of zeros and poles are fixed, whereas the actual values of the zeros and poles and their multiplicities are considered as variables.

Theorem by **Fuchs** and **Pethő**

Let n be a positive integer. Then there exists a positive integer $J \leq 2nn^{2n}$ and, for every $i \in \{1, \ldots, J\}$, an affine algebraic variety $\mathcal{V}_i$ defined over $\mathbb{Q}$ and with $\mathcal{V}_i \subset \mathbb{A}^{n+t_i}$ for some $2 \leq t_i \leq n$, such that:
(i) If $f, g, h \in k(x)$ with $f(x) = g(h(x))$ and with $\deg g, \deg h \geq 2$, g not of the shape $(\lambda(x))^m, m \in \mathbb{N}, \lambda \in \mathrm{PGL}_2(k)$, and f has at most n zeros and poles altogether, then there exists for some $i \in \{1, \ldots, J\}$ a point $P = (\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_{t_i}) \in \mathcal{V}_i(k)$,
a vector $(k_1, \ldots, k_{t_i}) \in \mathbb{Z}^{t_i}$ with $k_1 + k_2 + \ldots + k_{t_i} = 0$ or not depending on $\mathcal{V}_i$, a partition of $\{1, \ldots, n\}$ in $t_i + 1$ disjoint sets $S_\infty, S_{\beta_1}, \ldots, S_{\beta_{t_i}}$ with $S_\infty = \emptyset$ if $k_1 + k_2 + \cdots + k_{t_i} = 0$,
and a vector $(l_1, \ldots, l_n) \in \{0, 1, \ldots, n-1\}^n$, also both depending only on $\mathcal{V}_i$, such that

$$f(x)=\prod_{j=1}^{t_i}(w_j/w_\infty)^{k_j},\qquad g(x)=\prod_{j=1}^{t_i}(x-\beta_j)^{k_j},$$

and

$$h(x)=\begin{cases}\beta_j+\frac{w_j}{w_\infty}\ (j=1,\dots,t_i), & \text{if } k_1+k_2+\cdots+k_{t_i}\neq 0,\\ \frac{\beta_{j_1}w_{j_2}-\beta_{j_2}w_{j_1}}{w_{j_2}-w_{j_1}}\ (1\leq j_1<j_2\leq t_i), & \text{otherwise,}\end{cases}$$

where

$$w_j = \prod_{m\in S_{\beta_j}}(x-\alpha_m)^{l_m},\quad j=1,\dots,t_i,$$

$$w_\infty = \prod_{m\in S_\infty}(x-\alpha_m)^{l_m}.$$

Moreover, we have $\deg h\leq (n-1)/(t_i-1)\leq n-1$.

(ii) Conversely for given data $P \in \mathcal{V}_i(k), (k_1, \ldots, k_{t_i}), (l_1, \ldots, l_n)$, $S_\infty, S_{\beta_1} \ldots, S_{\beta_{t_i}}$, as described in (i) one defines by the same equations rational functions f, g, h with f having at most n zeros and poles altogether for which $f(x) = g(h(x))$ holds.

(iii) The integer J and equations defining the varieties $\mathcal{V}_i$ are effectively computable only in terms of n.

Tools from the theory of valuation

The **Mason-Stothers (1984) theorem** says: Let $f, g \in k(x)$, *not both constant and let* S *be any set of valuations of* $k(x)$ *containing all the zeros and poles in* $\mathbb{P}^1(k)$ *of* f *and* g*. Then we have* $\max\{\deg f, \deg g\} \leq |S| - 2.$ Best possible.

More generally **Zannier** (1995) proved: Let S is any set of valuations of $k(x)$ containing all the zeros and poles in $\mathbb{P}^1(k)$ of $g_1, \ldots, g_m$. If $g_1, \ldots, g_m \in k(x)$ *span a* k*-vector space of dimension* $\mu < m$ *and any* μ *of the* g_i *are linearly independent over* k*, then*

$$-\sum_{v \in \mathcal{M}} \min\{v(g_1), \ldots, v(g_m)\} \leq \frac{1}{m-\mu}\binom{\mu}{2}(|S|-2).$$

Since k is algebraically closed we can write

$$f(x) = \prod_{i=1}^{n} (x - \alpha_i)^{f_i}$$

with pairwise distinct $\alpha_i \in k$ and $f_i \in \mathbb{Z}$ for $i = 1, \ldots, n$.
Similarly we get

$$g(x) = \prod_{j=1}^{t} (x - \beta_j)^{k_j}$$

with pairwise distinct $\beta_j \in k$ and $k_j \in \mathbb{Z}$ for $j = 1, \ldots, t$ and $t \in \mathbb{N}$.
Thus we have

$$\prod_{i=1}^{n} (x - \alpha_i)^{f_i} = f(x) = g(h(x)) = \prod_{j=1}^{t} (h(x) - \beta_j)^{k_j}.$$

We now distinguish two cases depending on $k_1 + k_2 + \cdots + k_t \neq 0$ or not; observe that this condition is equivalent to $v_\infty(g) \neq 0$ or not.
We shall write $h(x) = p(x)/q(x)$ with $p, q \in k[x]$, p, q coprime.

The case $k_1 + k_2 + \cdots + k_t \neq 0$

There is a subset S_∞ of the set $\{1, \ldots, n\}$ such that the α_m for $m \in S_\infty$ are precisely the poles in $\mathbb{A}^1(k)$ of h, i.e.

$$q(x) = \prod_{m \in S_\infty} (x - \alpha_m)^{l_m}, \; l_m \in \mathbb{N}.$$

Furthermore h and $h(x) - \beta_j$ have the same number of poles counted by multiplicity, which means that their degrees are equal.

There is a partition of the set $\{1, \ldots, n\} \setminus S_\infty$ in t disjoint subsets $S_{\beta_1}, \ldots, S_{\beta_t}$ such that

$$h(x) = \beta_j + \frac{1}{q(x)} \prod_{m \in S_{\beta_j}} (x - \alpha_m)^{l_m},$$

where $l_m \in \mathbb{N}$ satisfies $l_m k_j = f_m$ for $m \in S_{\beta_j}, j = 1, \ldots, t$.

Since we assume that g is not of the shape $(\lambda(x))^m$ it follows that $t \geq 2$. Let $1 \leq i < j \leq t$ be given. We have at least two different representations of h and thus we get

$$\beta_i + \frac{1}{q(x)} \prod_{r \in S_{\beta_i}} (x - \alpha_r)^{l_r} = \beta_j + \frac{1}{q(x)} \prod_{s \in S_{\beta_j}} (x - \alpha_s)^{l_s}$$

or equivalently $\beta(u_i - u_j) = 1$, where $\beta = 1/(\beta_j - \beta_i)$ and

$$u_i = \frac{1}{q(x)} \prod_{r \in S_{\beta_i}} (x - \alpha_r)^{l_r} = \frac{w_i}{w_\infty}.$$

Actually, the u_i are S-units for the set of valuations $S = \{v_{\alpha_1}, \dots, v_{\alpha_n}, v_\infty\} \subset \mathcal{M}$ corresponding to $\alpha_1, \dots, \alpha_n \in k$ and ∞. In fact u_i and u_j have also no zeros in $\mathbb{A}^1(k)$ in common and they have all exactly the same poles (also with multiplicities), namely $\alpha_m, m \in S_\infty$ and possibly ∞.
The **Mason-Stothers** theorem implies that

$$l_m \leq n - 1 \text{ for all } m = 1, \dots, n.$$

We point out that the number of variables and the exponents depend only on n. Since $f(x) = g(h(x))$ is given at this point, there are k-rational points on this algebraic variety and one of them corresponds to $(\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_t)$ coming from f and g.

The case $k_1 + k_2 + \cdots + k_t = 0$

Here we have

$$\prod_{i=1}^{n}(x-\alpha_i)^{f_i} = \prod_{j=1}^{t}\left(\frac{p(x)}{q(x)} - \beta_j\right)^{k_j} = \prod_{j=1}^{t}(p(x)-\beta_j q(x))^{k_j}.$$

There is a partition of the set $\{1,\ldots,n\}$ in t disjoint subsets $S_{\beta_1},\ldots,S_{\beta_t}$ such that

$$(p(x)-\beta_j q(x))^{k_j} = \prod_{m\in S_{\beta_j}}(x-\alpha_m)^{f_m}.$$

Thus k_j divides f_m for all $m \in S_{\beta_j}, j = 1,\ldots,t$. On putting $l_m = f_m/k_j$ for $m \in S_{\beta_j}$ we obtain

$$p(x)-\beta_j q(x) = \prod_{m\in S_{\beta_j}}(x-\alpha_m)^{l_m}, j = 1,\ldots,t.$$

Let us choose $1 \leq j_1 < j_2 < j_3 \leq t$. From the corresponding three equations the so called **Siegel identity** $v_{j_1,j_2,j_3} + v_{j_3,j_1,j_2} + v_{j_2,j_3,j_1} = 0$ follows, where

$$v_{j_1,j_2,j_3} = (\beta_{j_1} - \beta_{j_2}) \prod_{m \in S_{\beta_{j_3}}} (x - \alpha_m)^{l_m}.$$

The quantities v_{j_1,j_2,j_3} are non-constant rational functions and they are S-units. Observe that by taking $j_1 = 1, j_2 = i, j_4 = j$ with $1 \leq i < j \leq t$ the Siegel identity can be rewritten as

$$\frac{\beta_j - \beta_1}{\beta_j - \beta_i} \frac{w_i}{w_1} + \frac{\beta_1 - \beta_i}{\beta_j - \beta_i} \frac{w_j}{w_1} = 1.$$

An algorithm to compute solutions

1) Let $S_\infty, S_{\beta_1}, \ldots, S_{\beta_t}$ be a partition of $\{1, 2, \ldots, n\}$.
2) For the partition and a vector $(l_1, \ldots, l_n) \in \{1, 2, \ldots, n\}^n$ compute the corresponding variety $V = \{v_1, \ldots, v_r\}$, where v_i is a polynomial in $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_t$. Here we used Groebner basis technique.
3) To remove contradictory systems we compute
$\Phi = \prod_{i \neq j}(\alpha_i - \alpha_j) \prod_{i \neq j}(\beta_i - \beta_j)$.
4) For all v_i compute

$$u_{i_1} = \frac{v_i}{\gcd(v_i, \Phi)},$$

and

$$u_{i_k} = \frac{u_{i_{k-1}}}{\gcd(u_{i_{k-1}}, \Phi)},$$

until $\gcd(u_{i_{k-1}}, \Phi) = 1$.

We performed the algorithm for $n = 3$ and $n = 4$ and obtained a complete list of all decomposable rational functions with number of singularities at most three or four. We have several sporadic examples for $n > 4$ too, but the number of partitions to be considered grows very fast, and we do not understand yet how to exclude very early the contradictory systems.

The case $t = 2, n = 3$ and $S_\infty \neq \emptyset$

There are two types of systems here, in the first class one obtains solutions having two parameters, in the second class one has solutions having three parameters. There are 18 systems which yield families with two parameters.

As an example consider the system from the sixth row, that is $(S_\infty, S_{\beta_1}, S_{\beta_2}) = (\{3\}, \{1\}, \{2\})$ and $(l_1, l_2, l_3) = (2, 1, 2)$. Here we obtain the following system of equations

$$\begin{aligned} \alpha_1 - \alpha_3 + 1/2 &= 0, \\ \alpha_2 - \alpha_3 + 1/4 &= 0, \\ \beta_1 - \beta_2 + 1 &= 0. \end{aligned}$$

Therefore one gets the parametric solution $(\alpha_3 - 1/2, \alpha_3 - 1/4, \alpha_3, \beta_2 - 1, \beta_2)$ and

$$f(x) = \frac{(x-\alpha_3+1/2)^2(x-\alpha_3+1/4)}{(x-\alpha_3)^4},$$
$$g(x) = (x-\beta_2+1)(x-\beta_2),$$
$$h(x) = \beta_2 - 1 + \frac{(x-\alpha_3+1/2)^2}{(x-\alpha_3)^2}.$$

There are 6 systems which yield families with three parameters.

$(S_\infty, S_{\beta_1}, S_{\beta_2}), (l_1, l_2, l_3)$	System of equations	Solution $(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2)$
$(\{3\}, \{2\}, \{1\})$ $(2, 2, 1)$	$\alpha_1 - \alpha_2 + 1/2\beta_1 - 1/2\beta_2 = 0$ $\alpha_2 - \alpha_3 - 1/4\beta_1 + 1/4\beta_2 = 0$	$(-\alpha_2 + 2\alpha_3, \alpha_2, \alpha_3, 4\alpha_2 - 4\alpha_3 + \beta_2, \beta_2)$
$(\{1\}, \{3\}, \{2\})$ $(1, 2, 2)$	$\alpha_1 - \alpha_3 + 1/4\beta_1 - 1/4\beta_2 = 0$ $\alpha_2 - \alpha_3 + 1/2\beta_1 - 1/2\beta_2 = 0$	$(\alpha_1, -\alpha_3 + 2\alpha_1, \alpha_3, -4\alpha_1 + 4\alpha_3 + \beta_2, \beta_2)$
$(\{2\}, \{3\}, \{1\})$ $(2, 1, 2)$	$\alpha_1 - \alpha_3 + 1/2\beta_1 - 1/2\beta_2 = 0$ $\alpha_2 - \alpha_3 + 1/4\beta_1 - 1/4\beta_2 = 0$	$(2\alpha_2 - \alpha_3, \alpha_2, \alpha_3, -4\alpha_2 + 4\alpha_3 + \beta_2, \beta_2)$
$(\{1\}, \{2\}, \{3\})$ $(1, 2, 2)$	$\alpha_1 - \alpha_3 - 1/4\beta_1 + 1/4\beta_2 = 0$ $\alpha_2 - \alpha_3 - 1/2\beta_1 + 1/2\beta_2 = 0$	$(\alpha_1, -\alpha_3 + 2\alpha_1, \alpha_3, 4\alpha_1 - 4\alpha_3 + \beta_2, \beta_2)$
$(\{3\}, \{1\}, \{2\})$ $(2, 2, 1)$	$\alpha_1 - \alpha_2 - 1/2\beta_1 + 1/2\beta_2 = 0$ $\alpha_2 - \alpha_3 + 1/4\beta_1 - 1/4\beta_2 = 0$	$(-\alpha_2 + 2\alpha_3, \alpha_2, \alpha_3, -4\alpha_2 + 4\alpha_3 + \beta_2, \beta_2)$
$(\{2\}, \{1\}, \{3\})$ $(2, 1, 2)$	$\alpha_1 - \alpha_3 - 1/2\beta_1 + 1/2\beta_2 = 0$ $\alpha_2 - \alpha_3 - 1/4\beta_1 + 1/4\beta_2 = 0$	$(2\alpha_2 - \alpha_3, \alpha_2, \alpha_3, 4\alpha_2 - 4\alpha_3 + \beta_2, \beta_2)$

Consider $(S_\infty, S_{\beta_1}, S_{\beta_2}) = (\{1\}, \{2\}, \{3\}), (l_1, l_2, l_3) = (1, 2, 2)$ and

$$\begin{aligned} \alpha_1 - \alpha_3 - 1/4\beta_1 + 1/4\beta_2 &= 0, \\ \alpha_2 - \alpha_3 - 1/2\beta_1 + 1/2\beta_2 &= 0. \end{aligned}$$

Thus

$$\begin{aligned} f(x) &= \frac{(x-\alpha_3)^2(x-2\alpha_1+\alpha_3)^2}{(x-\alpha_1)^2}, \\ g(x) &= (x-4\alpha_1+4\alpha_3-\beta_2)(x-\beta_2), \\ h(x) &= \beta_2 + \frac{(x-\alpha_3)^2}{x-\alpha_1}. \end{aligned}$$

The case $t = 3, n = 3$ and $S_\infty = \emptyset$

In total there are six parametrizations here, these are indicated in the table below.

$(S_{\beta_1}, S_{\beta_2}, S_{\beta_3},), (l_1, l_2, l_3)$	System of equations	Solution $(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3)$
$(\{1\}, \{3\}, \{2\})$ $(1, 1, 1)$	$\alpha_1\beta_2 - \alpha_1\beta_3 + \alpha_2\beta_1 - \alpha_2\beta_2 - \alpha_3\beta_1 + \alpha_3\beta_3 = 0$	$(-\frac{\alpha_2\beta_1-\alpha_2\beta_2-\alpha_3\beta_1+\alpha_3\beta_3}{\beta_2-\beta_3}, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3)$
$(\{2\}, \{1\}, \{3\})$ $(1, 1, 1)$	$\alpha_1\beta_1 - \alpha_1\beta_3 - \alpha_2\beta_2 + \alpha_2\beta_3 - \alpha_3\beta_1 + \alpha_3\beta_2 = 0$	$(\frac{\alpha_2\beta_2-\alpha_2\beta_3+\alpha_3\beta_1-\alpha_3\beta_2}{\beta_1-\beta_3}, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3)$
$(\{3\}, \{1\}, \{2\})$ $(1, 1, 1)$	$\alpha_1\beta_1 - \alpha_1\beta_3 - \alpha_2\beta_1 + \alpha_2\beta_2 - \alpha_3\beta_2 + \alpha_3\beta_3 = 0$	$(\frac{\alpha_2\beta_1-\alpha_2\beta_2+\alpha_3\beta_2-\alpha_3\beta_3}{\beta_1-\beta_3}, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3)$
$(\{1\}, \{2\}, \{3\})$ $(1, 1, 1)$	$\alpha_1\beta_2 - \alpha_1\beta_3 - \alpha_2\beta_1 + \alpha_2\beta_3 + \alpha_3\beta_1 - \alpha_3\beta_2 = 0$	$(\frac{\alpha_2\beta_1-\alpha_2\beta_3-\alpha_3\beta_1+\alpha_3\beta_2}{\beta_2-\beta_3}, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3)$
$(\{3\}, \{2\}, \{1\})$ $(1, 1, 1)$	$\alpha_1\beta_1 - \alpha_1\beta_2 - \alpha_2\beta_1 + \alpha_2\beta_3 + \alpha_3\beta_2 - \alpha_3\beta_3 = 0$	$(\frac{\alpha_2\beta_1-\alpha_2\beta_3-\alpha_3\beta_2+\alpha_3\beta_3}{\beta_1-\beta_2}, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3)$
$(\{2\}, \{3\}, \{1\})$ $(1, 1, 1)$	$\alpha_1\beta_1 - \alpha_1\beta_2 + \alpha_2\beta_2 - \alpha_2\beta_3 - \alpha_3\beta_1 + \alpha_3\beta_3 = 0$	$(-\frac{\alpha_2\beta_2-\alpha_2\beta_3-\alpha_3\beta_1+\alpha_3\beta_3}{\beta_1-\beta_2}, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3)$

As an illustration we provide an example corresponding to the parametrization indicated in the fourth row, that is $(S_{\beta_1}, S_{\beta_2}, S_{\beta_3}) = (\{1\}, \{2\}, \{3\})$ and $(l_1, l_2, l_3) = (1, 1, 1)$. Now let $(\alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3) = (2, 1, -1, 1, 0)$ and $k_1 = k_2 = 1, k_3 = -2$. One has that $\alpha_1 = 0$ and

$$
\begin{aligned}
f(x) &= \frac{(x-2)x}{(x-1)^2}, \\
g(x) &= \frac{(x-1)(x+1)}{x^2}, \\
h(x) &= x - 1.
\end{aligned}
$$

The case $t = 2, n = 4$ and $S_\infty \neq \emptyset$

There are 264 systems to deal with. We will treat only a few representative examples.

Systems containing two polynomials.
If $(S_\infty, S_{\beta_1}, S_{\beta_2}) = (\{4\}, \{1,2\}, \{3\})$ and $(l_1, l_2, l_3, l_4) = (1, 1, 2, 1)$, then we have

$$\begin{aligned} \alpha_1 + \alpha_2 - 2\alpha_3 - \beta_1 + \beta_2 &= 0 \\ \alpha_2^2 - 2\alpha_2\alpha_3 - \alpha_2\beta_1 + \alpha_2\beta_2 + \alpha_3^2 + \alpha_4\beta_1 - \alpha_4\beta_2 &= 0. \end{aligned}$$

Since $\alpha_i \neq \alpha_j$ and $\beta_i \neq \beta_j$ if $i \neq j$, we have that

$$\begin{aligned} \alpha_1 &= -\alpha_2 + 2\alpha_3 + \beta_1 - \beta_2, \\ \alpha_4 &= \alpha_2 - \frac{(\alpha_2 - \alpha_3)^2}{\beta_1 - \beta_2}. \end{aligned}$$

For example, if we consider the solution $(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta_1, \beta_2) = (-2, 1, 0, 2, 0, 1)$, then we get

$$\begin{aligned} f(x) &= \frac{(x-1)x^2(x+2)}{(x-2)^2}, \\ g(x) &= (x-1)x, \\ h(x) &= \frac{(x-1)(x+2)}{x-2}. \end{aligned}$$

Systems containing three polynomials.
If $(S_\infty, S_{\beta_1}, S_{\beta_2}) = (\{1\}, \{2, 3\}, \{4\})$ and $(l_1, l_2, l_3, l_4) = (1, 2, 1, 3)$, then we get

$$\begin{aligned} \alpha_1 + 1/3\alpha_3 - 4/3\alpha_4 &= 0 \\ \alpha_2 + 1/2\alpha_3 - 3/2\alpha_4 &= 0 \\ \alpha_3^2 - 2\alpha_3\alpha_4 + \alpha_4^2 - 4/3\beta_1 + 4/3\beta_2 &= 0. \end{aligned}$$

Thus one obtains the parametrization

$$\begin{aligned} \alpha_1 &= -1/3\alpha_3 + 4/3\alpha_4, \\ \alpha_2 &= -1/2\alpha_3 + 3/2\alpha_4, \\ \beta_1 &= 3/4\alpha_3^2 - 3/2\alpha_3\alpha_4 + 3/4\alpha_4^2 + \beta_2. \end{aligned}$$

Let us take $(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta_1, \beta_2) = (-1/3, -1/2, 1, 0, 1, 1/4)$, then we have

$$\begin{aligned} f(x) &= \frac{(x-1)x^3(x+1/2)^2}{(x+1/3)^2}, \\ g(x) &= (x-1)(x-1/4), \\ h(x) &= \frac{1}{4} + \frac{x^3}{x+1/3}. \end{aligned}$$

Systems containing four polynomials.
Consider the case $(S_\infty, S_{\beta_1}, S_{\beta_2}) = (\{1\},\{2,3\},\{4\})$ and $(l_1, l_2, l_3, l_4) = (3,1,1,3)$. One gets the system

$$
\begin{aligned}
\alpha_1 - \alpha_4 - 1/3 &= 0\\
\alpha_2 + \alpha_3 - 2\alpha_4 - 1/3 &= 0\\
\alpha_3^2 - 2\alpha_3\alpha_4 - 1/3\alpha_3 + \alpha_4^2 + 1/3\alpha_4 + 1/27 &= 0\\
\beta_1 - \beta_2 - 1 &= 0.
\end{aligned}
$$

The parametrization is as follows

$$
\begin{aligned}
\alpha_1 &= \alpha_4 + 1/3,\\
\alpha_2 &= \alpha_4 \mp \frac{\sqrt{-3}}{18} + \frac{1}{6},\\
\alpha_3 &= \alpha_4 \pm \frac{\sqrt{-3}}{18} + \frac{1}{6},\\
\beta_1 &= \beta_2 + 1.
\end{aligned}
$$

As an example we take $(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta_1, \beta_2) = (1/6, -\sqrt{-3}/18, \sqrt{-3}/18, -1/6, 1, 0)$, then we obtain

$$
\begin{aligned}
f(x) &= \frac{(x - \sqrt{-3}/18)(x + \sqrt{-3}/18)(x + 1/6)^3}{(x - 1/6)^6},\\
g(x) &= (x-1)x,\\
h(x) &= \frac{(x + 1/6)^3}{(x - 1/6)^3}.
\end{aligned}
$$

Systems containing five polynomials.
If $(S_\infty, S_{\beta_1}, S_{\beta_2}) = (\{1\}, \{2,3\}, \{4\})$ and $(l_1, l_2, l_3, l_4) = (3,1,2,2)$, then we have

$$\begin{aligned}
\alpha_1 - 1/3\alpha_2 - 2/3\alpha_3 - 1/3 &= 0\\
\alpha_2^2 - 2\alpha_2\alpha_4 + 2\alpha_2 + 8\alpha_3^2 - 16\alpha_3\alpha_4 + 6\alpha_3 + 9\alpha_4^2 - 8\alpha_4 + 1 &= 0\\
\alpha_2 + 7/2\alpha_3 - 9/2\alpha_4 + 1 &= 0\\
\alpha_3 - \alpha_4 + 8/27 &= 0\\
\beta_1 - \beta_2 + 1 &= 0.
\end{aligned}$$

As a concrete example we deal with the case $(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta_1, \beta_2) = (4/27, 1/27, -8/27, 0, 0, 1)$. It easily follows that

$$\begin{aligned}
f(x) &= \frac{(x-1/27)x^2(x+8/27)^2}{(x-4/27)^6},\\
g(x) &= (x-1)x,\\
h(x) &= 1 + \frac{x^2}{(x-4/27)^3}.
\end{aligned}$$

The case $t = 3, n = 4$ and $S_\infty \neq \emptyset$

We could not eliminate 24 systems directly by our approach, but it turned out that in any possible solution either $\alpha_i = \alpha_j$ for some $i \neq j$, or $\beta_i = \beta_j$ for some $i \neq j$. Let us consider a concrete example out of the 24 systems. If $(S_\infty, S_{\beta_1}, S_{\beta_2}, S_{\beta_3}) = (\{2\}, \{1\}, \{3\}, \{4\})$ and $(l_1, l_2, l_3, l_4) = (2, 2, 2, 1)$, then we have

$$\begin{aligned}
\alpha_1 - \alpha_4 + 1/4 &= 0, \\
\alpha_2 - \alpha_4 - 1/4 &= 0, \\
\alpha_3 - \alpha_4 + 1/4 &= 0, \\
\beta_1 - \beta_3 + 1 &= 0, \\
\beta_2 - \beta_3 + 1 &= 0.
\end{aligned}$$

The last two equations yield that $\beta_1 = \beta_2$, a contradiction.

The case $t = 3, n = 4$ and $S_\infty = \emptyset$

There are 6 systems having two polynomials in the Groebner basis, one of these is as follows: $(S_{\beta_1}, S_{\beta_2}, S_{\beta_3}) = (\{1,3\},\{4\},\{2\})$ and $(l_1, l_2, l_3, l_4) = (1,2,1,2)$.
As an example consider the case $(\alpha_2, \alpha_3, \alpha_4, \beta_1, \beta_3) = (0,1,3,0,1)$.
We obtain that $\alpha_1 = -3$ and $\beta_2 = 4$. Let $k_1 = k_2 = 1$ and $k_3 = -2$.
We get that

$$\begin{aligned} f(x) &= \frac{(x-3)^2(x-1)(x+3)}{x^4}, \\ g(x) &= \frac{(x-4)x}{(x-1)^2}, \\ h(x) &= \frac{(x-1)(x+3)}{2x-3}. \end{aligned}$$

There are 18 systems having three polynomials in the Groebner basis, one of these is as follows: $(S_{\beta_1}, S_{\beta_2}, S_{\beta_3}) = (\{1\}, \{2, 3\}, \{4\})$ and $(l_1, l_2, l_3, l_4) = (2, 1, 1, 2)$.
Now we consider the example with $(\alpha_2, \alpha_3, \alpha_4, \beta_1, \beta_2) = (0, 1, 3, 0, 1)$.
We have that $\alpha_1 = 2/3$ and $\beta_3 = -8$. Let $k_1 = k_2 = 1$ and $k_3 = -2$.
We have that

$$\begin{aligned} f(x) &= \frac{(x-2/3)^2(x-1)x}{(x-2)^4}, \\ g(x) &= \frac{(x-1)x}{(x+8)^2}, \\ h(x) &= \frac{(3x-2)^2}{-3x+4}. \end{aligned}$$

The case $t = 4, n = 4$ and $S_\infty = \emptyset$

Here we have 24 systems to solve. Since one has 24 very similar systems, we will deal with one of these only. Let $(S_{\beta_1}, S_{\beta_2}, S_{\beta_3}, S_{\beta_4}) = (\{1\}, \{2\}, \{3\}, \{4\})$ and $(l_1, l_2, l_3, l_4) = (1, 1, 1, 1)$.
Now let $(\alpha_3, \alpha_4, \beta_1, \beta_2, \beta_3, \beta_4) = (0, 1, 3, 2, 1, 0)$ and $k_1 = k_2 = 1, k_3 = k_4 = -1$. One obtains that

$$\begin{aligned} f(x) &= \frac{(x+1)(x+2)}{(x-1)x}, \\ g(x) &= \frac{(x-3)(x-2)}{(x-1)x}, \\ h(x) &= -x+1. \end{aligned}$$