ON COMPOSITE RATIONAL FUNCTIONS

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Abstract. In this paper we characterize all composite lacunary rational functions having at most 4 distinct zeroes and poles and we also provide some examples in case of 5 singularities.

1. introduction

In this article we deal with a problem related to decompositions of polynomials and rational functions. A classical result by Ritt [28] states that if there is a polynomial $f \in \mathbb{C}[X]$ satisfying certain tameness properties and

$$f = g_1 \circ g_2 \circ \cdots \circ g_r = h_1 \circ h_2 \circ \cdots \circ h_s,$$

then $r = s$ and $\{\deg g_1, \ldots, \deg g_r\} = \{\deg h_1, \ldots, \deg h_r\}$. Ritt's fundamental result has been investigated, extended and applied in various wide-ranging contexts (see e.g. [5, 11, 13, 14, 17, 18, 21, 22, 24, 25, 32, 33]). The above statement is not true for rational functions. It is not true that all complete decompositions of a rational function have the same length. Gutierrez and Sevilla [21] provided an example with rational coefficients as follows

$$f = \frac{x^3(x+6)^3(x^2-6x+36)^3}{(x-3)^3(x^2+3x+9)^3},$$

$$f = g_1 \circ g_2 \circ g_3 = x^3 \circ \frac{x(x-12)}{x-3} \circ \frac{x(x+6)}{x-3},$$

$$f = h_1 \circ h_2 = \frac{x^3(x+24)}{x-3} \circ \frac{x(x^2-6x+36)}{x^2+3x+9}.$$

We would like to emphasize that combinations of Siegel's [30] and Faltings' [16] finiteness theorems, related to integral and rational points on algebraic curves, and Ritt's result have yielded many nice results in Diophantine number theory (see e.g. [2, 6, 8, 7, 10, 12, 18, 23, 26, 27, 31]).

In case of lacunary polynomials, that is when the number of terms of the polynomial is considered to be fixed while the degrees and coefficients may vary, Erdős [15] and independently Rényi posed the following conjecture. If $h(x)^2$ has boundedly many terms, then the same is true for $h(x) \in \mathbb{C}[X]$. Schinzel [29] gave a proof in a more general case, namely when $h(x)^d$ has boundedly many terms. Schinzel made the conjecture that if $g(h(x))$ has boundadly many term, then it holds also for $h(x)$. This latter conjecture has been proved by Zannier [34]. Fuchs and Zannier [20] extended the problem, they considered lacunary rational functions which are decomposable. An other possibility to think about lacunarity is that one considers the number of zeros and poles of a rational function in reduced form to be bounded. In this case Fuchs and Pethő [19] obtained results related to the structure of such decomposable rational functions. We note that their proof was algorithmic, in this paper we provide some computational experiments that we obtained by using

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a MAGMA [9] implementation. We not only compute the appropriate varieties, but we also provide parametrizations of the possible solutions. We remark that algorithms have been developed earlier to find decompositions of a given rational function (see e.g. [1, 3, 4]). In [3], Ayad and Fleischmann implemented a MAGMA code to find decompositions, as an example they considered the rational function

$$f = \frac{x^4 - 8x}{x^3 + 1}$$

and they obtained that $f(x) = g(h(x))$, where

$$g = \frac{x^2 + 4x}{x + 1} \quad \text{and} \quad h = \frac{x^2 - 2x}{x + 1}.$$

2. AUXILIARY RESULTS

Fuchs and Pethő [19] proved the following theorem.

Theorem A. *Let n be a positive integer. Then there exists a positive integer J and, for every $i \in \{1, \ldots, J\}$, an affine algebraic variety V_i defined over $\mathbb{Q}$ and with $V_i \subset \mathbb{A}^{n+t_i}$ for some $2 \leq t_i \leq n$, such that:*

(i) If $f, g, h \in k(x)$ with $f(x) = g(h(x))$ and with $\deg g, \deg h \geq 2, g$ not of the shape $(\lambda(x))^m, m \in \mathbb{N}, \lambda \in PGL_2(k)$, and f has at most n zeros and poles altogether, then there exists for some $i \in \{1, \ldots, J\}$ a point $P = (\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_{t_i}) \in V_i(k)$, a vector $(k_1, \ldots, k_{t_i}) \in \mathbb{Z}^{t_i}$ with $k_1 + k_2 + \ldots + k_{t_i} = 0$ or not depending on V_i , a partition of $\{1, \ldots, n\}$ in $t_i + 1$ disjoint sets $S_\infty, S_{\beta_1}, \ldots, S_{\beta_{t_i}}$ with $S_\infty = \emptyset$ if $k_1 + k_2 + \ldots + k_{t_i} = 0$, and a vector $(l_1, \ldots, l_n) \in \{0, 1, \ldots, n-1\}^n$, also both depending only on V_i, such that

$$f(x) = \prod_{j=1}^{t_i} (\omega_j/\omega_\infty)^{k_j}, \quad g(x) = \prod_{j=1}^{t_i} (x - \beta_j)^{k_j}$$

and

$$h(x) = \begin{cases} \beta_j + \frac{\omega_j}{\omega_\infty} \quad (j = 1, \ldots, t_i), & \text{if } k_1 + k_2 + \ldots + k_{t_i} \neq 0 \\ \frac{\beta_{j_1}\omega_{j_2} - \beta_{j_2}\omega_{j_1}}{\omega_{j_2} - \omega_{j_1}} \quad (1 \leq j_1 < j_2 \leq t_i), & \text{otherwise,} \end{cases}$$

where

$$\omega_j = \prod_{m \in S_{\beta_j}} (x - \alpha_m)^{l_m}, \quad j = 1, \ldots, t_i$$

and

$$\omega_\infty = \prod_{m \in S_\infty} (x - \alpha_m)^{l_m}.$$

Moreover, we have $degh \leq (n-1)/\max t_i - 2, 1 \leq n - 1$.

(ii) Conversely for given data $P \in V_i(k), (k_1, \ldots, k_{t_i}), S_\infty, S_{\beta_1}, \ldots, S_{\beta_{t_i}}, (l_1, \ldots, l_n)$ as described in (i) one defines by the same equations rational functions f, g, h with f having at most n zeros and poles altogether for which $f(x) = g(h(x))$ holds.

(iii) The integer J and equations defining the varieties V_i are effectively computable only in terms of n.

The method of proof of the above Theorem is effective. It provides an algorithm to obtain all possible decompositions of rational functions with at most n zeros and poles altogether.

We introduce some notation. Let

$$f(x) = \prod_{i=1}^{n} (x - \alpha_i)^{f_i}$$

with pairwise distinct $\alpha_i \in k$ and $f_i \in \mathbb{Z}$ for $i = 1, \ldots, n$. (Remember that without loss of generality we are assuming that f is monic.) Similarly, let

$$g(x) = \prod_{j=1}^{t} (x - \beta_j)^{k_j}$$

with pairwise distinct $\beta_j \in k$ and $k_j \in \mathbb{Z}$ for $j = 1, \ldots, t$ and $t \in \mathbb{N}$. Therefore we have

$$\prod_{i=1}^{n} (x - \alpha_i)^{f_i} = f(x) = g(h(x)) = \prod_{j=1}^{t} (h(x) - \beta_j)^{k_j}.$$

We shall write $h(x) = p(x)/q(x)$ with $p, q \in k[x], p, q$ coprime. Fuchs and Pethő showed that if $S_\infty \neq \emptyset$ then

$$q(x) = \prod_{m \in S_\infty} (x - \alpha_m)^{l_m}$$

and there is a partition of the set $\{1, \ldots, n\} \setminus S_\infty$ in t disjoint subsets $S_{\beta_1}, \ldots, S_{\beta_t}$ such that

$$h(x) = \beta_j + \frac{1}{q(x)} \prod_{m \in S_j} (x - \alpha_m)^{l_m}, \tag{1}$$

where $l_m \in \mathbb{N}$ satisfies $l_m k_j = f_m$ for $m \in S_{\beta_j}$, and this holds true for every $j = 1, \ldots, t$. We obtain at least two different representations of h, since we assumed that g is not of the shape $(\lambda(x))^m$. Hence we get at least one equation of the form

$$\beta_i + \frac{1}{q(x)} \prod_{r \in S_i} (x - \alpha_r)^{l_r} = \beta_j + \frac{1}{q(x)} \prod_{s \in S_j} (x - \alpha_s)^{l_s}. \tag{2}$$

If $S_\infty = \emptyset$ then we have

$$(p(x) - \beta_j q(x))^{k_j} = \prod_{m \in S_{\beta_j}} (x - \alpha_m)^{f_m}.$$

Now we have that $t \geq 3$, otherwise g is in the special form we excluded. Siegel's identity provides the equations in this case. That is if $1 \leq j_1 < j_2 < j_3 \leq t$, then we have

$$v_{j_1,j_2,j_3} + v_{j_3,j_1,j_2} + v_{j_2,j_3,j_1} = 0, \tag{3}$$

where

$$v_{j_1,j_2,j_3} = (\beta_{j_1} - \beta_{j_2}) \prod_{m \in S_{\beta_{j_3}}} (x - \alpha_m)^{l_m}.$$

3. THE COMPUTATION

The method of proof by Fuchs and Pethő provides an algorithm to obtain the possible varieties. So we followed the steps described below.

(1) compute the partitions of $\{1, 2, \ldots, n\}$ into $t + 1$ disjoint sets
(2) given a partition $S_\infty, S_{\beta_1}, \ldots, S_{\beta_t}$ and a vector $(l_1, \ldots, l_n) \in \{1, 2, \ldots, n\}^n$ compute the corresponding variety $V = \{v_1, \ldots, v_r\}$, where v_i is a polynomial in $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_t$ obtained from (2) or (3)
(3) compute Groebner basis V_G of the ideal generated by the polynomials $v_1, \ldots, v_r$
(4) test ideal membership for all $\alpha_i - \alpha_j, i, j = 1, 2, \ldots, n, i \neq j$ and $\beta_i - \beta_j, i, j = 1, 2, \ldots, t, i \neq j$
(5) if there is no contradiction in the system list the given partition, vector and variety.

Now we provide some details of the computation. We note that there are no solution if $t=3, n=3$ and $S_\infty \neq \emptyset$ or $t=3, n=4$ and $S_\infty \neq \emptyset$ or $t=4, n=4$ and $S_\infty \neq \emptyset$.

3.1. **The case $t=2, n=3$ and $S_\infty \neq \emptyset$.** There are two types of systems here, in the first class one obtains solutions having two parameters, in the second class one has solutions having three parameters. Below we indicate the 18 systems which yield families with two parameters.

$(S_\infty, S_{\beta_1}, S_{\beta_2}), (l_1, l_2, l_3)$	System of equations	Solution $(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2)$
$(\{3\},\{2\},\{1\})$ $(1,2,2)$	$\alpha_1-\alpha_3+1/4=0$ $\alpha_2-\alpha_3+1/2=0$ $\beta_1-\beta_2+1=0$	$(-1/4+\alpha_3, -1/2+\alpha_3, \alpha_3, -1+\beta_2, \beta_2)$
$(\{2\},\{1\},\{3\})$ $(2,2,1)$	$\alpha_1-\alpha_3+1/4=0$ $\alpha_2-\alpha_3-1/4=0$ $\beta_1-\beta_2+1=0$	$(-1/4+\alpha_3, 1/4+\alpha_3, \alpha_3, -1+\beta_2, \beta_2)$
$(\{1\},\{2\},\{3\})$ $(2,2,1)$	$\alpha_1-\alpha_3-1/4=0$ $\alpha_2-\alpha_3+1/4=0$ $\beta_1-\beta_2+1=0$	$(1/4+\alpha_3, -1/4+\alpha_3, \alpha_3, -1+\beta_2, \beta_2)$
$(\{1\},\{3\},\{2\})$ $(2,1,2)$	$\alpha_1-\alpha_3-1/2=0$ $\alpha_2-\alpha_3-1/4=0$ $\beta_1-\beta_2+1=0$	$(1/2+\alpha_3, 1/4+\alpha_3, \alpha_3, -1+\beta_2, \beta_2)$
$(\{2\},\{3\},\{1\})$ $(2,2,1)$	$\alpha_1-\alpha_3+1/4=0$ $\alpha_2-\alpha_3-1/4=0$ $\beta_1-\beta_2-1=0$	$(-1/4+\alpha_3, 1/4+\alpha_3, \alpha_3, 1+\beta_2, \beta_2)$
$(\{3\},\{1\},\{2\})$ $(2,1,2)$	$\alpha_1-\alpha_3+1/2=0$ $\alpha_2-\alpha_3+1/4=0$ $\beta_1-\beta_2+1=0$	$(-1/2+\alpha_3, -1/4+\alpha_3, \alpha_3, -1+\beta_2, \beta_2)$
$(\{1\},\{3\},\{2\})$ $(2,2,1)$	$\alpha_1-\alpha_3-1/4=0$ $\alpha_2-\alpha_3+1/4=0$ $\beta_1-\beta_2-1=0$	$(1/4+\alpha_3, -1/4+\alpha_3, \alpha_3, 1+\beta_2, \beta_2)$
$(\{1\},\{2\},\{3\})$ $(2,1,2)$	$\alpha_1-\alpha_3-1/2=0$ $\alpha_2-\alpha_3-1/4=0$ $\beta_1-\beta_2-1=0$	$(1/2+\alpha_3, 1/4+\alpha_3, \alpha_3, 1+\beta_2, \beta_2)$
$(\{2\},\{1\},\{3\})$ $(1,2,2)$	$\alpha_1-\alpha_3-1/4=0$ $\alpha_2-\alpha_3-1/2=0$ $\beta_1-\beta_2-1=0$	$(1/4+\alpha_3, 1/2+\alpha_3, \alpha_3, 1+\beta_2, \beta_2)$
$(\{2\},\{3\},\{1\})$ $(1,2,2)$	$\alpha_1-\alpha_3-1/4=0$ $\alpha_2-\alpha_3-1/2=0$ $\beta_1-\beta_2+1=0$	$(1/4+\alpha_3, 1/2+\alpha_3, \alpha_3, -1+\beta_2, \beta_2)$
$(\{3\},\{2\},\{1\})$ $(2,1,2)$	$\alpha_1-\alpha_3+1/2=0$ $\alpha_2-\alpha_3+1/4=0$ $\beta_1-\beta_2-1=0$	$(-1/2+\alpha_3, -1/4+\alpha_3, \alpha_3, 1+\beta_2, \beta_2)$
$(\{3\},\{1\},\{2\})$ $(1,2,2)$	$\alpha_1-\alpha_3+1/4=0$ $\alpha_2-\alpha_3+1/2=0$ $\beta_1-\beta_2-1=0$	$(-1/4+\alpha_3, -1/2+\alpha_3, \alpha_3, 1+\beta_2, \beta_2)$

As an example consider the system from the sixth row, that is $(S_\infty, S_{\beta_1}, S_{\beta_2}) = (\{3\},\{1\},\{2\})$ and $(l_1, l_2, l_3) = (2,1,2)$. Here we obtain the following system of equations

$$\begin{aligned} \alpha_1-\alpha_3+1/2 &= 0, \\ \alpha_2-\alpha_3+1/4 &= 0, \\ \beta_1-\beta_2+1 &= 0. \end{aligned}$$

Therefore one gets the parametric solution $(\alpha_3-1/2, \alpha_3-1/4, \alpha_3, \beta_2-1, \beta_2)$ and

$$\begin{aligned} f(x) &= \frac{(x-\alpha_3+1/2)^2(x-\alpha_3+1/4)}{(x-\alpha_3)^4}, \\ g(x) &= (x-\beta_2+1)(x-\beta_2), \\ h(x) &= \beta_2-1+\frac{(x-\alpha_3+1/2)^2}{(x-\alpha_3)^2}. \end{aligned}$$

We note that one gets the same family in case of $(S_\infty, S_{\beta_1}, S_{\beta_2}) = (\{3\},\{2\},\{1\})$ and $(l_1,l_2,l_3)=(1,2,2)$.

Now we provide the table containing the 6 systems which yield families with three parameters.

$(S_\infty, S_{\beta_1}, S_{\beta_2}),(l_1,l_2,l_3)$	System of equations	Solution $(\alpha_1,\alpha_2,\alpha_3,\beta_1,\beta_2)$
$(\{3\},\{2\},\{1\})$ $(2,2,1)$	$\alpha_1-\alpha_2+1/2\beta_1-1/2\beta_2=0$ $\alpha_2-\alpha_3-1/4\beta_1+1/4\beta_2=0$	$(-\alpha_2+2\alpha_3,\alpha_2,\alpha_3,4\alpha_2-4\alpha_3+\beta_2,\beta_2)$
$(\{1\},\{3\},\{2\})$ $(1,2,2)$	$\alpha_1-\alpha_3+1/4\beta_1-1/4\beta_2=0$ $\alpha_2-\alpha_3+1/2\beta_1-1/2\beta_2=0$	$(\alpha_1,-\alpha_3+2\alpha_1,\alpha_3,-4\alpha_1+4\alpha_3+\beta_2,\beta_2)$
$(\{2\},\{3\},\{1\})$ $(2,1,2)$	$\alpha_1-\alpha_3+1/2\beta_1-1/2\beta_2=0$ $\alpha_2-\alpha_3+1/4\beta_1-1/4\beta_2=0$	$(2\alpha_2-\alpha_3,\alpha_2,\alpha_3,-4\alpha_2+4\alpha_3+\beta_2,\beta_2)$
$(\{1\},\{2\},\{3\})$ $(1,2,2)$	$\alpha_1-\alpha_3-1/4\beta_1+1/4\beta_2=0$ $\alpha_2-\alpha_3-1/2\beta_1+1/2\beta_2=0$	$(\alpha_1,-\alpha_3+2\alpha_1,\alpha_3,4\alpha_1-4\alpha_3+\beta_2,\beta_2)$
$(\{3\},\{1\},\{2\})$ $(2,2,1)$	$\alpha_1-\alpha_2-1/2\beta_1+1/2\beta_2=0$ $\alpha_2-\alpha_3+1/4\beta_1-1/4\beta_2=0$	$(-\alpha_2+2\alpha_3,\alpha_2,\alpha_3,-4\alpha_2+4\alpha_3+\beta_2,\beta_2)$
$(\{2\},\{1\},\{3\})$ $(2,1,2)$	$\alpha_1-\alpha_3-1/2\beta_1+1/2\beta_2=0$ $\alpha_2-\alpha_3-1/4\beta_1+1/4\beta_2=0$	$(2\alpha_2-\alpha_3,\alpha_2,\alpha_3,4\alpha_2-4\alpha_3+\beta_2,\beta_2)$

From the parametrizations one can easily obtain the corresponding rational functions, as an example we take the fourth row of the table. That is, we have $(S_\infty, S_{\beta_1}, S_{\beta_2}) = (\{1\},\{2\},\{3\}), (l_1,l_2,l_3)=(1,2,2)$ and

$$\begin{aligned} \alpha_1-\alpha_3-1/4\beta_1+1/4\beta_2 &= 0, \\ \alpha_2-\alpha_3-1/2\beta_1+1/2\beta_2 &= 0. \end{aligned}$$

Thus

$$\begin{aligned} f(x) &= \frac{(x-\alpha_3)^2(x-2\alpha_1+\alpha_3)^2}{(x-\alpha_1)^2}, \\ g(x) &= (x-4\alpha_1+4\alpha_3-\beta_2)(x-\beta_2), \\ h(x) &= \beta_2+\frac{(x-\alpha_3)^2}{x-\alpha_1}. \end{aligned}$$

3.2. **The case $t=3, n=3$ and $S_\infty=\emptyset$.** In total there are six parametrizations here, these are indicated in the table below.

$(S_{\beta_1}, S_{\beta_2}, S_{\beta_3},), (l_1, l_2, l_3)$	System of equations	Solution $(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3)$
$(\{1\}, \{3\}, \{2\})$ $(1, 1, 1)$	$\alpha_1\beta_2 - \alpha_1\beta_3 + \alpha_2\beta_1 - \alpha_2\beta_2 - \alpha_3\beta_1 + \alpha_3\beta_3 = 0$	$(-\frac{\alpha_2\beta_1-\alpha_2\beta_2-\alpha_3\beta_1+\alpha_3\beta_3}{\beta_2-\beta_3}, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3)$
$(\{2\}, \{1\}, \{3\})$ $(1, 1, 1)$	$\alpha_1\beta_1 - \alpha_1\beta_3 - \alpha_2\beta_2 + \alpha_2\beta_3 - \alpha_3\beta_1 + \alpha_3\beta_2 = 0$	$(\frac{\alpha_2\beta_2-\alpha_2\beta_3+\alpha_3\beta_1-\alpha_3\beta_2}{\beta_1-\beta_3}, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3)$
$(\{3\}, \{1\}, \{2\})$ $(1, 1, 1)$	$\alpha_1\beta_1 - \alpha_1\beta_3 - \alpha_2\beta_1 + \alpha_2\beta_2 - \alpha_3\beta_2 + \alpha_3\beta_3 = 0$	$(\frac{\alpha_2\beta_1-\alpha_2\beta_2+\alpha_3\beta_2-\alpha_3\beta_3}{\beta_1-\beta_3}, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3)$
$(\{1\}, \{2\}, \{3\})$ $(1, 1, 1)$	$\alpha_1\beta_2 - \alpha_1\beta_3 - \alpha_2\beta_1 + \alpha_2\beta_3 + \alpha_3\beta_1 - \alpha_3\beta_2 = 0$	$(\frac{\alpha_2\beta_1-\alpha_2\beta_3-\alpha_3\beta_1+\alpha_3\beta_2}{\beta_2-\beta_3}, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3)$
$(\{3\}, \{2\}, \{1\})$ $(1, 1, 1)$	$\alpha_1\beta_1 - \alpha_1\beta_2 - \alpha_2\beta_1 + \alpha_2\beta_3 + \alpha_3\beta_2 - \alpha_3\beta_3 = 0$	$(\frac{\alpha_2\beta_1-\alpha_2\beta_3-\alpha_3\beta_2+\alpha_3\beta_3}{\beta_1-\beta_2}, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3)$
$(\{2\}, \{3\}, \{1\})$ $(1, 1, 1)$	$\alpha_1\beta_1 - \alpha_1\beta_2 + \alpha_2\beta_2 - \alpha_2\beta_3 - \alpha_3\beta_1 + \alpha_3\beta_3 = 0$	$(-\frac{\alpha_2\beta_2-\alpha_2\beta_3-\alpha_3\beta_1+\alpha_3\beta_3}{\beta_1-\beta_2}, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3)$

As an illustration we provide an example corresponding to the parametrization indicated in the fourth row, that is $(S_{\beta_1}, S_{\beta_2}, S_{\beta_3}) = (\{1\}, \{2\}, \{3\})$ and $(l_1, l_2, l_3) = (1, 1, 1)$. Now let $(\alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3) = (2, 1, -1, 1, 0)$ and $k_1 = k_2 = 1, k_3 = -2$. One has that $\alpha_1 = 0$ and

$$\begin{aligned} f(x) &= \frac{(x-2)x}{(x-1)^2}, \\ g(x) &= \frac{(x-1)(x+1)}{x^2}, \\ h(x) &= x-1. \end{aligned}$$

3.3. **The case $t = 2, n = 4$ and $S_\infty \neq \emptyset$.** There are 264 systems to deal with. We will treat only a few representative examples.

Systems containing two polynomials.

If $(S_\infty, S_{\beta_1}, S_{\beta_2}) = (\{4\}, \{1, 2\}, \{3\})$ and $(l_1, l_2, l_3, l_4) = (1, 1, 2, 1)$, then we have

$$\begin{aligned} \alpha_1 + \alpha_2 - 2\alpha_3 - \beta_1 + \beta_2 &= 0 \\ \alpha_2^2 - 2\alpha_2\alpha_3 - \alpha_2\beta_1 + \alpha_2\beta_2 + \alpha_3^2 + \alpha_4\beta_1 - \alpha_4\beta_2 &= 0. \end{aligned}$$

Since $\alpha_i \neq \alpha_j$ and $\beta_i \neq \beta_j$ if $i \neq j$, we have that

$$\begin{aligned} \alpha_1 &= -\alpha_2 + 2\alpha_3 + \beta_1 - \beta_2, \\ \alpha_4 &= \alpha_2 - \frac{(\alpha_2 - \alpha_3)^2}{\beta_1 - \beta_2}. \end{aligned}$$

For example, if we consider the solution $(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta_1, \beta_2) = (-2, 1, 0, 2, 0, 1)$, then we get

$$\begin{aligned} f(x) &= \frac{(x-1)x^2(x+2)}{(x-2)^2}, \\ g(x) &= (x-1)x, \\ h(x) &= \frac{(x-1)(x+2)}{x-2}. \end{aligned}$$

Systems containing three polynomials.

If $(S_\infty, S_{\beta_1}, S_{\beta_2}) = (\{1\}, \{2, 3\}, \{4\})$ and $(l_1, l_2, l_3, l_4) = (1, 2, 1, 3)$, then we get

$$\begin{aligned} \alpha_1 + 1/3\alpha_3 - 4/3\alpha_4 &= 0 \\ \alpha_2 + 1/2\alpha_3 - 3/2\alpha_4 &= 0 \\ \alpha_3^2 - 2\alpha_3\alpha_4 + \alpha_4^2 - 4/3\beta_1 + 4/3\beta_2 &= 0. \end{aligned}$$

Thus one obtains the parametrization

$$\begin{aligned}\alpha_1 &= -1/3\alpha_3 + 4/3\alpha_4,\\ \alpha_2 &= -1/2\alpha_3 + 3/2\alpha_4,\\ \beta_1 &= 3/4\alpha_3^2 - 3/2\alpha_3\alpha_4 + 3/4\alpha_4^2 + \beta_2.\end{aligned}$$

Let us take $(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta_1, \beta_2) = (-1/3, -1/2, 1, 0, 1, 1/4)$, then we have

$$\begin{aligned}f(x) &= \frac{(x-1)x^3(x+1/2)^2}{(x+1/3)^2},\\ g(x) &= (x-1)(x-1/4),\\ h(x) &= \frac{1}{4} + \frac{x^3}{x+1/3}.\end{aligned}$$

Systems containing four polynomials.
Consider the case $(S_\infty, S_{\beta_1}, S_{\beta_2}) = (\{1\}, \{2, 3\}, \{4\})$ and $(l_1, l_2, l_3, l_4) = (3, 1, 1, 3)$. One gets the system

$$\begin{aligned}\alpha_1 - \alpha_4 - 1/3 &= 0\\ \alpha_2 + \alpha_3 - 2\alpha_4 - 1/3 &= 0\\ \alpha_3^2 - 2\alpha_3\alpha_4 - 1/3\alpha_3 + \alpha_4^2 + 1/3\alpha_4 + 1/27 &= 0\\ \beta_1 - \beta_2 - 1 &= 0.\end{aligned}$$

The parametrization is as follows

$$\begin{aligned}\alpha_1 &= \alpha_4 + 1/3,\\ \alpha_2 &= \alpha_4 \mp \frac{\sqrt{-3}}{18} + \frac{1}{6},\\ \alpha_3 &= \alpha_4 \pm \frac{\sqrt{-3}}{18} + \frac{1}{6},\\ \beta_1 &= \beta_2 + 1.\end{aligned}$$

As an example we take $(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta_1, \beta_2) = (1/6, -\sqrt{-3}/18, \sqrt{-3}/18, -1/6, 1, 0)$, then we obtain

$$\begin{aligned}f(x) &= \frac{(x-\sqrt{-3}/18)(x+\sqrt{-3}/18)(x+1/6)^3}{(x-1/6)^6},\\ g(x) &= (x-1)x,\\ h(x) &= \frac{(x+1/6)^3}{(x-1/6)^3}.\end{aligned}$$

Systems containing five polynomials.
If $(S_\infty, S_{\beta_1}, S_{\beta_2}) = (\{1\}, \{2, 3\}, \{4\})$ and $(l_1, l_2, l_3, l_4) = (3, 1, 2, 2)$, then we have

$$\begin{aligned}\alpha_1 - 1/3\alpha_2 - 2/3\alpha_3 - 1/3 &= 0\\ \alpha_2^2 - 2\alpha_2\alpha_4 + 2\alpha_2 + 8\alpha_3^2 - 16\alpha_3\alpha_4 + 6\alpha_3 + 9\alpha_4^2 - 8\alpha_4 + 1 &= 0\\ \alpha_2 + 7/2\alpha_3 - 9/2\alpha_4 + 1 &= 0\\ \alpha_3 - \alpha_4 + 8/27 &= 0\\ \beta_1 - \beta_2 + 1 &= 0.\end{aligned}$$

We get the parametrization

$$\begin{aligned}\alpha_1 &= \alpha_4 + 4/27,\\ \alpha_2 &= \alpha_4 + 1/27,\\ \alpha_3 &= \alpha_4 - 8/27,\\ \beta_1 &= \beta_2 - 1.\end{aligned}$$

As a concrete example we deal with the case $(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\beta_1,\beta_2)=(4/27,1/27,-8/27,0,0,1)$. It easily follows that

$$\begin{aligned} f(x) &= \frac{(x-1/27)x^2(x+8/27)^2}{(x-4/27)^6},\\ g(x) &= (x-1)x,\\ h(x) &= 1+\frac{x^2}{(x-4/27)^3}. \end{aligned}$$

3.4. **The case $t=3, n=4$ and $S_\infty=\emptyset$.** There are 24 systems to handle in this case. The systems are getting more and more complicated therefore we deal with two typical cases. There are 6 systems having two polynomials in the Groebner basis, one of these is as follows: $(S_{\beta_1},S_{\beta_2},S_{\beta_3})=(\{1,3\},\{4\},\{2\})$ and $(l_1,l_2,l_3,l_4)=(1,2,1,2)$. The system of equations are given by

$$\begin{aligned} \alpha_1\beta_2-\alpha_1\beta_3+2\alpha_2\beta_1-2\alpha_2\beta_2+\alpha_3\beta_2-\alpha_3\beta_3-2\alpha_4\beta_1+2\alpha_4\beta_3 &= 0\\ \alpha_2^2\beta_1-\alpha_2^2\beta_2-2\alpha_2\alpha_3\beta_1+2\alpha_2\alpha_3\beta_2-\alpha_3^2\beta_2+\alpha_3^2\beta_3+2\alpha_3\alpha_4\beta_1-2\alpha_3\alpha_4\beta_3-\alpha_4^2\beta_1+\alpha_4^2\beta_3 &= 0. \end{aligned}$$

There are four solutions where $\alpha_i=\alpha_j$ or $\beta_i=\beta_j$

$$\begin{gathered} (\alpha_1=\alpha_4,\alpha_2=\alpha_4,\alpha_3=\alpha_4,\alpha_4,\beta_1,\beta_2,\beta_3),\\ (\alpha_1=\alpha_3,\alpha_2=\alpha_3,\alpha_3,\alpha_4,\beta_1=\beta_3,\beta_2,\beta_3),\\ (\alpha_1,\alpha_2,\alpha_3,\alpha_4,\beta_1=\beta_3,\beta_2=\beta_3,\beta_3),\\ (\alpha_1,\alpha_2=\alpha_4,\alpha_3,\alpha_4,\beta_1,\beta_2=\beta_3,\beta_3). \end{gathered}$$

These solutions do not lead to appropriate rational functions. There is one solution which yield solutions of the original problem

$$\begin{aligned} \alpha_1 &= -\frac{\alpha_2\alpha_3-2\alpha_2\alpha_4+\alpha_3\alpha_4}{\alpha_2-2\alpha_3+\alpha_4},\\ \beta_2 &= \frac{\alpha_2^2\beta_1-2\alpha_2\alpha_3\beta_1+\alpha_3^2\beta_3+2\alpha_3\alpha_4\beta_1-2\alpha_3\alpha_4\beta_3-\alpha_4^2\beta_1+\alpha_4^2\beta_3}{(\alpha_2-\alpha_3)^2}, \end{aligned}$$

where $\alpha_2,\alpha_3,\alpha_4,\beta_1,\beta_3$ are parameters such that $\alpha_i\neq\alpha_j,\beta_i\neq\beta_j$ and $\alpha_2-2\alpha_3+\alpha_4\neq 0$. As an example consider the case $(\alpha_2,\alpha_3,\alpha_4,\beta_1,\beta_3)=(0,1,3,0,1)$. We obtain that $\alpha_1=-3$ and $\beta_2=4$. Let $k_1=k_2=1$ and $k_3=-2$. We get that

$$\begin{aligned} f(x) &= \frac{(x-3)^2(x-1)(x+3)}{x^4},\\ g(x) &= \frac{(x-4)x}{(x-1)^2},\\ h(x) &= \frac{(x-1)(x+3)}{2x-3}. \end{aligned}$$

There are 18 systems having three polynomials in the Groebner basis, one of these is as follows: $(S_{\beta_1},S_{\beta_2},S_{\beta_3})=(\{1\},\{2,3\},\{4\})$ and $(l_1,l_2,l_3,l_4)=(2,1,1,2)$. The system of equations is

$$\begin{aligned} \alpha_1\alpha_2+\alpha_1\alpha_3-2\alpha_1\alpha_4-2\alpha_2\alpha_3+\alpha_2\alpha_4+\alpha_3\alpha_4 &= 0\\ \alpha_1\beta_2-\alpha_1\beta_3-1/2\alpha_2\beta_1+1/2\alpha_2\beta_3-1/2\alpha_3\beta_1+1/2\alpha_3\beta_3+\alpha_4\beta_1-\alpha_4\beta_2 &= 0\\ \alpha_2^2\beta_1-\alpha_2^2\beta_3+2\alpha_2\alpha_3\beta_1-4\alpha_2\alpha_3\beta_2+2\alpha_2\alpha_3\beta_3-4\alpha_2\alpha_4\beta_1+ &\\ +4\alpha_2\alpha_4\beta_2+\alpha_3^2\beta_1-\alpha_3^2\beta_3-4\alpha_3\alpha_4\beta_1+4\alpha_3\alpha_4\beta_2+4\alpha_4^2\beta_1-4\alpha_4^2\beta_2 &= 0. \end{aligned}$$

The only solution where one can obtain appropriate rational functions is

$$\begin{aligned} \alpha_1 &= \frac{-\alpha_2\alpha_4-\alpha_3\alpha_4+2\alpha_2\alpha_3}{\alpha_2+\alpha_3-2\alpha_4},\\ \beta_3 &= \frac{\alpha_2^2\beta_1+2\alpha_2\alpha_3\beta_1-4\alpha_2\alpha_3\beta_2-4\alpha_2\alpha_4\beta_1+4\alpha_2\alpha_4\beta_2+\alpha_3^2\beta_1-4\alpha_3\alpha_4\beta_1+4\alpha_3\alpha_4\beta_2+4\alpha_4^2\beta_1-4\alpha_4^2\beta_2}{(\alpha_2-\alpha_3)^2}, \end{aligned}$$

where $\alpha_2, \alpha_3, \alpha_4, \beta_1, \beta_3$ are parameters such that $\alpha_i \neq \alpha_j, \beta_i \neq \beta_j$ and $\alpha_2 + \alpha_3 - 2\alpha_4 \neq 0$. Now we consider the example with $(\alpha_2, \alpha_3, \alpha_4, \beta_1, \beta_2) = (0, 1, 3, 0, 1)$. We have that $\alpha_1 = 2/3$ and $\beta_3 = -8$. Let $k_1 = k_2 = 1$ and $k_3 = -2$. We have that

$$\begin{aligned} f(x) &= \frac{(x-2/3)^2(x-1)x}{(x-2)^4}, \\ g(x) &= \frac{(x-1)x}{(x+8)^2}, \\ h(x) &= \frac{(3x-2)^2}{-3x+4}. \end{aligned}$$

3.5. **The case $t = 4, n = 4$ and $S_\infty = \emptyset$.** Here we have 24 systems to solve. Since one has 24 very similar systems, we will deal with one of these only. Let $(S_{\beta_1}, S_{\beta_2}, S_{\beta_3}, S_{\beta_4}) = (\{1\}, \{2\}, \{3\}, \{4\})$ and $(l_1, l_2, l_3, l_4) = (1, 1, 1, 1)$. One gets the system of equations

$$\begin{aligned} \alpha_1\beta_2 - \alpha_1\beta_4 - \alpha_2\beta_1 + \alpha_2\beta_4 + \alpha_4\beta_1 - \alpha_4\beta_2 &= 0 \\ \alpha_1\beta_3 - \alpha_1\beta_4 - \alpha_3\beta_1 + \alpha_3\beta_4 + \alpha_4\beta_1 - \alpha_4\beta_3 &= 0 \\ \alpha_2\beta_3 - \alpha_2\beta_4 - \alpha_3\beta_2 + \alpha_3\beta_4 + \alpha_4\beta_2 - \alpha_4\beta_3 &= 0. \end{aligned}$$

There are three solutions which do not correspond to appropriate rational functions, the remaining solution has

$$\begin{aligned} \alpha_1 &= \frac{\alpha_3\beta_1 - \alpha_3\beta_4 - \alpha_4\beta_1 + \alpha_4\beta_3}{\beta_3 - \beta_4}, \\ \alpha_2 &= \frac{\alpha_3\beta_2 - \alpha_3\beta_4 - \alpha_4\beta_2 + \alpha_4\beta_3}{\beta_3 - \beta_4}. \end{aligned}$$

Now let $(\alpha_3, \alpha_4, \beta_1, \beta_2, \beta_3, \beta_4) = (0, 1, 3, 2, 1, 0)$ and $k_1 = k_2 = 1, k_3 = k_4 = -1$. One obtains that

$$\begin{aligned} f(x) &= \frac{(x+1)(x+2)}{(x-1)x}, \\ g(x) &= \frac{(x-3)(x-2)}{(x-1)x}, \\ h(x) &= -x+1. \end{aligned}$$

4. TWO EXAMPLES WITH $n = 5$

We computed all the varieties corresponding to $n = 5$, the systems are getting more and more complicated therefore we selected only two examples given below. Some details of the computations can be found in the following table and all systems in case of $n = 5$ can be downloaded from
`http://www.math.unideb.hu/~tengely/CFunc5.txt.tar.gz`

n	t	S_∞	# systems
5	2	$\neq \emptyset$	4644
5	3	$\neq \emptyset$	60
5	4	$\neq \emptyset$	0
5	5	$\neq \emptyset$	0
5	3	$\emptyset$	384
5	4	$\emptyset$	0
5	5	$\emptyset$	120

In this section we provide two examples in case of $n = 5$.

- Consider the case $(S_\infty, S_{\beta_1}, S_{\beta_2}) = (\{1,5\},\{3,4\},\{2\})$ and $(l_1,l_2,l_3,l_4,l_5) = (3,1,1,3,1)$. One gets a system containing 5 equations

$$\begin{aligned} \alpha_1 - 2\alpha_4 + \alpha_5 &= 0 \\ \alpha_2 - 3/2\alpha_4 + 1/2\alpha_5 &= 0 \\ \alpha_3 - 3\alpha_4 + 2\alpha_5 &= 0 \\ \alpha_4^3 - 3\alpha_4^2\alpha_5 + 3\alpha_4\alpha_5^2 - \alpha_5^3 + 1/2 &= 0 \\ \beta_1 - \beta_2 + 1 &= 0. \end{aligned}$$

The solutions of this system of equations are given by

$$(\alpha_1, \alpha_1 + \frac{1}{4}\sqrt[3]{4}\zeta^k, \frac{1}{2}(\sqrt[3]{2}\alpha_1 - 1)\sqrt[3]{4}\zeta^k, \frac{1}{2}(\sqrt[3]{2}\alpha_1 + 1)\sqrt[3]{4}\zeta^k, \frac{1}{2}(\sqrt[3]{2}\alpha_1 + 2)\sqrt[3]{4}\zeta^k, \beta_1, \beta_1 + 1),$$

where $\zeta = \frac{1+i\sqrt{3}}{2}$ and $k = 0,1,2$.

- Let $(S_\infty, S_{\beta_1}, S_{\beta_2}) = (\{1,2,5\},\{3\},\{4\})$ and $(l_1,l_2,l_3,l_4,l_5) = (1,1,1,3,1)$. We obtain the following system of equations

$$\begin{aligned} \alpha_1 + \alpha_2 - 3\alpha_4 + \alpha_5 &= 0 \\ \alpha_2^2 - 3\alpha_2\alpha_4 + \alpha_2\alpha_5 + 3\alpha_4^2 - 3\alpha_4\alpha_5 + \alpha_5^2 - 1 &= 0 \\ \alpha_3 - \alpha_4^3 + 3\alpha_4^2\alpha_5 - 3\alpha_4\alpha_5^2 + \alpha_5^3 - \alpha_5 &= 0 \\ \beta_1 - \beta_2 - 1 &= 0 \end{aligned}$$

The general solutions are given by

$$\begin{aligned} &\alpha_1, \\ &\alpha_2, \\ &\alpha_3 = \frac{1}{18}\sqrt{-\alpha_1+\alpha_2+2}\left(4\sqrt{\alpha_1-\alpha_2+2}\sqrt{3}\alpha_1\alpha_2 - 2\sqrt{\alpha_1-\alpha_2+2}\sqrt{3}\alpha_2^2 - \sqrt{\alpha_1-\alpha_2+2}\left(2\sqrt{3}\alpha_1^2+\sqrt{3}\right)\right) + \\ &\frac{1}{2}\alpha_1 + \frac{1}{2}\alpha_2, \\ &\alpha_4 = -\frac{1}{6}\sqrt{-\alpha_1^2+2\alpha_1\alpha_2-\alpha_2^2+4}\sqrt{3} + \frac{1}{2}\alpha_1 + \frac{1}{2}\alpha_2, \\ &\alpha_5 = -\frac{1}{2}\sqrt{-\alpha_1+\alpha_2+2}\sqrt{\alpha_1-\alpha_2+2}\sqrt{3} + \frac{1}{2}\alpha_1 + \frac{1}{2}\alpha_2 \\ &\beta_1, \\ &\beta_2 = \beta_1 - 1. \end{aligned}$$

and

$$\begin{aligned} &\alpha_1, \\ &\alpha_2, \\ &\alpha_3 = -\frac{1}{18}\sqrt{-\alpha_1+\alpha_2+2}\left(4\sqrt{\alpha_1-\alpha_2+2}\sqrt{3}\alpha_1\alpha_2 - 2\sqrt{\alpha_1-\alpha_2+2}\sqrt{3}\alpha_2^2 - \sqrt{\alpha_1-\alpha_2+2}\left(2\sqrt{3}\alpha_1^2+\sqrt{3}\right)\right) + \\ &\frac{1}{2}\alpha_1 + \frac{1}{2}\alpha_2, \\ &\alpha_4 = \frac{1}{6}\sqrt{-\alpha_1^2+2\alpha_1\alpha_2-\alpha_2^2+4}\sqrt{3} + \frac{1}{2}\alpha_1 + \frac{1}{2}\alpha_2, \\ &\alpha_5 = \frac{1}{2}\sqrt{-\alpha_1+\alpha_2+2}\sqrt{\alpha_1-\alpha_2+2}\sqrt{3} + \frac{1}{2}\alpha_1 + \frac{1}{2}\alpha_2, \\ &\beta_1, \\ &\beta_2 = \beta_1 - 1. \end{aligned}$$

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