POWER VALUES OF SUMS OF CERTAIN PRODUCTS OF CONSECUTIVE INTEGERS AND RELATED RESULTS - SUPPLEMENTARY MATERIAL

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1. Rational solutions of the equation $y^2 = g_T(x)$ with $T \in A_n, n \leq 5$.

Let $\mathbb{N}$ denote the set of positive integers, $\mathbb{N}_0$ the set of non-negative integers and $\mathbb{N}_{\geq k}$ will denote the set of non-negative integers $\geq k$. For $n \in \mathbb{N}_0$ we write

$$p_a(x) = \prod_{i=0}^{a}(x+j).$$

Moreover, we define the set

$$A_n = \{(a_1, \ldots, a_k) \in \mathbb{N}_0^k : \ a_i < a_{i+1} \text{ for } i = 1, 2, \ldots k-1, \ a_k < n \text{ and } k \in \{1, \ldots, n-1\}\}.$$

For given $m \in \mathbb{N}_{\geq 2}$ and $T = (a_1, \ldots, a_k) \in A_n$ we consider the Diophantine equation

$$(1) \qquad y^m = g_T(x), \quad \text{where} \quad g_T(x) := p_n(x) + \sum_{i=1}^{k} p_{a_i}(x).$$

Let $n \in \mathbb{N}$ and for given $T \in A_n$ let us consider the algebraic curve

$$C_T : \ y^2 = g_T(x).$$

Let us write $gen(T) := \text{genus}(C_T)$ - the genus of the curve C_T and $J_T := \text{Jac}(C_T)$ - the Jacobian variety associated with C_T. Moreover, we define $r(T) := \text{rank}(J_T)$ - the rank of the Jacobian variety J_T. As usual, by $C_T(\mathbb{Q})$ we will denote the set of all rational points on the curve C_T and by $C_T(\mathbb{Z})$ - the set of integral points on C_T.

In this section we consider the Diophantine equation (1) for $T \in A_n$, with $n \leq 5$. We present complete system of integer solutions in the considered cases and where it is possible we compute the set of all *rational* solutions. This is particulary interesting in the cases when $gen(T) \geq 2$ because in this case the set $C_T(\mathbb{Q})$ is finite by Faltings theorem [15].

We start with the following.

Theorem 1.1. *Let $T \in A_2$.*

1. *If $T = (0)$, then $C_T(\mathbb{Q}) = \{(0,0), \mathcal{O}\}$.*
2. *If $T = (1)$, then $C_T(\mathbb{Q}) = \{(-3,0), (-1,0), (0,0), \mathcal{O}\}$.*
3. *If $T = (0,1)$, then $C_T(\mathbb{Q}) = \{(t^2, t(t^2+2)) : \ t \in \mathbb{Q}\} \cup \{\mathcal{O}\}$.*

Proof. If $T = (0)$, then $g_T(x) = x(x^2+3x+3)$. The corresponding curve C_T has rank 0 and the torsion group consisting only two points: the finite point $(0,0)$ and the point at infinity.

If $T = (1)$, then $g_T(x) = x(x+1)(x+3)$. The corresponding curve C_T has rank 0 and the torsion group isomorphic with $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. The only rational points on C_T are exactly those from the statement.

Finally, if $T = (0,1)$, then $g_T(x) = x(x+2)^2$ and the result is clear. $\square$

Theorem 1.2. *Let $T \in A_3$.*

1. *If $gen(T) = 0$, then $T \in \{(0,1), (1,2)\}$. We have*

$$C_{(0,1)}(\mathbb{Q}) = \left\{\left(\frac{2}{t^2-1}, \frac{4t^3}{(t^2-1)^2}\right) : \ t \in \mathbb{Q} \setminus \{\pm 1\}\right\} \text{ and } C_{(0,1)}(\mathbb{Z}) = \{(-2,0), (0,0)\},$$

$$C_{(1,2)}(\mathbb{Q}) = \left\{\left(\frac{1}{t^2-1}, \frac{t(3t^2-2)}{(t^2-1)^2}\right) : \ t \in \mathbb{Q} \setminus \{\pm 1\}\right\} \text{ and } C_{(1,2)}(\mathbb{Z}) = \{(-1,0), (0,0)\}$$

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(2) *If $T \in A_3 \setminus \{(0,1),(1,2)\}$, then $gen(T) = 1, r(T) = 1$, and the set $C_T(\mathbb{Q})$ is infinite. Moreover, we have the following:*
   (a) *if $T = (0)$, then $C_T(\mathbb{Z}) = \{(1, \pm 5), (0, 0)\}$;*
   (b) *if $T = (1)$, then $C_T(\mathbb{Z}) = \{(-1, 0), (-4, \pm 6), (0, 0)\}$;*
   (c) *if $T = (2)$, then $C_T(\mathbb{Z}) = \{(-2, 0), (-1, 0), (2, \pm 12), (-4, 0), (0, 0)\}$;*
   (d) *if $T = (0, 2)$, then $C_T(\mathbb{Z}) = \{(0, 0)\}$;*
   (e) *if $T = (0, 1, 2)$, then $C_T(\mathbb{Z}) = \{(-2, 0), (0, 0)\}$.*

Proof. If $T = (0,1),(1,2)$ we have $g_T(x) = x(x+2)^3$ and $g_T(x) = x(1+x)(3+x)^2$ respectively. In particular $gen(T) = 0$. Using a standard method of parametrization of quadratic curves we get the description of the set of rational points on the corresponding curves.

In all remaining cases we deal with genus 1 curve of the form $y^2 = g_T(x)$, where g_T is a monic quartic polynomial of degree 4. In particular, the equation under consideration satisfies Runge's condition [22] and a simple application of standard methods give the full description of $C_T(\mathbb{Z})$ presented in the statement. The rank of the corresponding elliptic curve was computed with the help of Magma [6]. □

Theorem 1.3. *Let $T \in A_4$.*

(1) *If $gen(T) = 0$, then $T = (0)$ and $C_T(\mathbb{Q}) = \{(t^2, t(t^4 + 5t^2 + 5)) : t \in \mathbb{Q}\}$.*
(2) *If $gen(T) = 1$, then $T \in \{(1, 2), (2, 3)\}$. We have $r(T) = 0$ and*

$$C_{(1,2)}(\mathbb{Q}) = \{(0, 0), (-1, 0), (-3, 0), \mathcal{O}\},$$
$$C_{(2,3)}(\mathbb{Q}) = \{(0, 0), (-1, 0), (-2, 0), \mathcal{O}\}.$$

(3) *If $T \in A_4 \setminus \{(0), (1, 2), (2, 3)\}$, then $gen(T) = 2$ and $r(T) \leq 1$. We collect the data concerning the set $C_T(\mathbb{Q})$ in the table below. In the description of the set $C_T(\mathbb{Q})$ we omit the point at infinity.*

T	$r(T)$	$C_T(\mathbb{Q})$
(1)	0	$\{(-1, 0), (0, 0)\}$
(2)	0	$\{(-2, 0), (-1, 0), (0, 0)\}$
(3)	1	$\{(-5, 0), (-3, 0), (-1, 0), (0, 0), (1, \pm 12)\}$
$(0, 1)$	0	$\{(-2, 0), (0, 0)\}$
$(0, 2)$	1	$\{(0, 0)\}$
$(0, 3)$	0	$\{(0, 0)\}$
$(1, 3)$	1	$\{(-4, \pm 6), (-1, 0), (0, 0)\}$
$(0, 1, 2)$	0	$\{(-2, 0), (0, 0)\}$
$(0, 1, 3)$	0	$\{(-2, 0), (0, 0)\}$
$(0, 2, 3)$	1	$\{(0, 0)\}$
$(1, 2, 3)$	0	$\{(-3, 0), (-1, 0), (0, 0)\}$
$(0, 1, 2, 3)$	0	$\{(-2, 0), (0, 0)\}$

Table 1. Rational points on the genus two curves C_T.

Proof. For $T = (0)$ we have $g_T(x) = x(x^2 + 5x + 5)^2$ and the description of the set $C_T(\mathbb{Q})$ is clear.

For $T = (1, 2), (2, 3)$, we have $g_T(x) = x(x + 1)(x + 3)^3$ and $g_T(x) = x(x + 1)(x + 2)(x + 4)^2$. In both cases the corresponding elliptic curve is of rank 0. The torsion points are the only integer points on the corresponding curves.

For the remaining values of $T \in A_4$, the corresponding curve C_T is of genus 2. Fortunately, in each case, the rank of the Jacobian variety J_T associated with C_T is bounded by 1. Thus in each case we can apply Chabauty's method [10] in order to find complete set of rational points on the curve C_T. The procedures in case of genus 2 curves were implemented in Magma based on papers by Stoll [24, 25, 26]. More precisely, in case of $r_T = 0$ we can use directly the following commands:

```
A<x>:=PolynomialRing(Rationals());
C:=HyperellipticCurve(f(x));
J:=Jacobian(C);
Chabauty0(C);
```

,

where f is our polynomial of degree 5 or 6 without multiple roots and such that rank of J is equal to 0. The computation of bound for the rank is performed with the procedure `RankBound(J)`.

The case when the rank of the Jacobian is equal to 1, the situation is a bit different. We first apply the procedure `Points(J: Bound:=`10^3`)` in order to find the set, say A, of all rational divisors on the Jacobian J with the height bounded by 10^3. The chosen bound is grater then the $e^{h(J)}$, where $h(J)$ is the height constant associated with the Jacobian under consideration. Then, we compute the reduced basis, say R, with the help of the procedure `ReducedBasis(A)`. Next, we look for divisors of infinite order in R, by checking the order of the elements of R with the help of procedure `Order(P)` for each $P \in R$. Finally, for the set B of all divisors of infinite order we apply the procedure `Chabauty(B)` and get the required set of all rational points on C. □

We were unable to characterize rational solutions on the curve C_T for all $T \in A_5$. The partial result is contained in the following:

Theorem 1.4. *Let* $T \in A_5$.

(1) *If* $g(T) = 0$, *then* $T = (1)$ *and*

$$C_T(\mathbb{Q}) = \left\{ \left(\frac{t^2}{1-2t}, \frac{(t-1)t(t^4 - 14t^3 + 51t^2 - 44t + 11)}{(1-2t)^3} \right) : t \in \mathbb{Q} \setminus \{1/2\} \right\}$$

and $C_Q(\mathbb{Z}) = \{(-1,0),(0,0)\}$.

(2) *If* $g(T) = 1$, *then* $T = (2,3)$ *or* $T = (3,4)$. *If* $T = (3,4)$, *then* $r(T) = 0$ *and*

$$C_{(3,4)}(\mathbb{Q}) = C_{(3,4)}(\mathbb{Z}) = \{(-5,0),(-3,0),(-2,0),(-1,0),(0,0)\}.$$

If $T = (2,3)$, *then* $r(T) = 1$ *and the set* $C(\mathbb{Q})$ *is infinite.*

(3) *If* $g(T) = 2$ *and* $\mathrm{rk}(T) \le 1$, *then the values of* T *and the corresponding set of rational points are as follows. In the description of the set* $C_T(\mathbb{Q})$ *we omit the points at infinity* $(1, \pm 1, 0)$.

T	$r(T)$	$C_T(\mathbb{Q})$
(0)	1	$\{(0,0)\}$
(4)	1	$\{(-6,0),(-4,0),(-3,0),(-2,0),(-1,0),(0,0),(-12/7,\pm 720/7)\}$
$(0,1)$	1	$\{(-2,0),(0,0)\}$
$(0,2)$	1	$\{(0,0)\}$
$(0,3)$	1	$\{(0,0)\}$
$(1,2)$	1	$\{(-3,0),(-1,0),(0,0)\}$
$(2,4)$	1	$\{(-2,0),(-1,0),(0,0)\}$
$(0,1,3)$	1	$\{(-2,0),(0,0)\}$
$(0,1,4)$	1	$\{(-2,0),(0,0)\}$
$(0,2,4)$	1	$\{(0,0)\}$
$(0,3,4)$	1	$\{(121/120,\pm 51334697/120),(0,0)\}$
$(1,2,4)$	1	$\{(-3,0),(-1,0),(0,0)\}$
$(2,3,4)$	1	$\{(-38/11,\pm 1368/11),(-4,0),(-2,0),(0,0)\}$
$(0,1,2,4)$	1	$\{(-2,0),(0,0)\}$
$(0,1,3,4)$	1	$\{(-2,0),(-25/8,\pm 1335/8),(0,0)\}$
$(0,2,3,4)$	1	$\{(0,0)\}$
$(1,2,3,4)$	1	$\{(-3,0),(-1,0),(0,0)\}$

Table 3. Rational points on the genus two curves C_T for $T \in A_5$ and $r(T) \le 1$

Proof. If $T = (1)$, then $g_T(x) = p_1(x) + p_5(x) = x(x+1)(x^2+7x+11)^2$. The rational parametrization of the curve $v^2 = x(x+1)$ is given by

$$x = \frac{t^2}{1-2t}, \quad v = \frac{(1-t)t}{1-2t}$$

and hence the result (with $y = v(x^2 + 7x + 11)$).

If $T = (3,4)$, then $g_T(x) = p_3(x)+p_4(x)+p_5(x) = x(x+1)(x+2)(x+3)(x+5)^2$. The curve $v^2 = x(x+1)(x+2)(x+3)$ is birationally equivalent with the elliptic curve $E_T:\ y^2 = x^3+11x^2+36x+36$ of rank 0 and $\mathrm{Tors}(E) \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$.

If $T = (2,3)$, then $g_T(x) = p_2(x) + p_3(x) + p_5(x) = x(x+1)(x+2)(x+4)^3$. The curve $v^2 = x(x+1)(x+2)(x+4)$ is birationally equivalent with the elliptic curve $E_T:\ y^2 = x^3+14x^2+56x+64$ of rank 1, where the point of infinite order is $P = (0,-8)$ and $\mathrm{Tors}(E) \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$.

In remaining cases of $T \in A_5$, we get that the curve C_T is of genus 2 with $r(T) = 1$ (as computed by MAGMA). Thus in each case we can apply Chabauty's method and get the result. □

2. Application of Runge method for several equations $y^m = g_T(x)$.

Theorem 2.1. *For $T \in A_5$ consider the curve $C_T:\ y^2 = g_T(x)$. The complete list of integral solutions are given in the Table 2.*

T	*polynomial*	$C_T(\mathbb{Z})$
(0)	$(1, 15, 85, 225, 274, 121, 0)$	$\{(0, 0)\}$
(1)	$(1, 15, 85, 225, 275, 121, 0)$	$\{(-1, 0), (0, 0)\}$
(2)	$(1, 15, 85, 226, 277, 122, 0)$	$\{(-1, 0), (-2, 0), (0, 0)\}$
(3)	$(1, 15, 86, 231, 285, 126, 0)$	$\{(-1, 0), (-9, \pm 252), (-2, 0), (-3, 0), (0, 0)\}$
(4)	$(1, 16, 95, 260, 324, 144, 0)$	$\{(-4, 0), (-1, 0), (-2, 0), (-6, 0), (-3, 0), (0, 0)\}$
$(0, 1)$	$(1, 15, 85, 225, 275, 122, 0)$	$\{(-2, 0), (0, 0)\}$
$(0, 2)$	$(1, 15, 85, 226, 277, 123, 0)$	$\{(0, 0)\}$
$(0, 3)$	$(1, 15, 86, 231, 285, 127, 0)$	$\{(0, 0)\}$
$(0, 4)$	$(1, 16, 95, 260, 324, 145, 0)$	$\{(1, \pm 29), (0, 0)\}$
$(1, 2)$	$(1, 15, 85, 226, 278, 123, 0)$	$\{(-1, 0), (-3, 0), (0, 0)\}$
$(1, 3)$	$(1, 15, 86, 231, 286, 127, 0)$	$\{(-4, \pm 6), (-1, 0), (0, 0)\}$
$(1, 4)$	$(1, 16, 95, 260, 325, 145, 0)$	$\{(-1, 0), (0, 0)\}$
$(2, 3)$	$(1, 15, 86, 232, 288, 128, 0)$	$\{(-4, 0), (2, \pm 72), (-1, 0), (-2, 0), (0, 0)\}$
$(2, 4)$	$(1, 16, 95, 261, 327, 146, 0)$	$\{(-1, 0), (-2, 0), (0, 0)\}$
$(3, 4)$	$(1, 16, 96, 266, 335, 150, 0)$	$\{(-1, 0), (-5, 0), (-2, 0), (-3, 0), (0, 0)\}$
$(0, 1, 2)$	$(1, 15, 85, 226, 278, 124, 0)$	$\{(1, \pm 27), (-2, 0), (0, 0)\}$
$(0, 1, 3)$	$(1, 15, 86, 231, 286, 128, 0)$	$\{(-2, 0), (0, 0)\}$
$(0, 1, 4)$	$(1, 16, 95, 260, 325, 146, 0)$	$\{(-2, 0), (0, 0)\}$
$(0, 2, 3)$	$(1, 15, 86, 232, 288, 129, 0)$	$\{(0, 0)\}$
$(0, 2, 4)$	$(1, 16, 95, 261, 327, 147, 0)$	$\{(0, 0)\}$
$(0, 3, 4)$	$(1, 16, 96, 266, 335, 151, 0)$	$\{(0, 0)\}$
$(1, 2, 3)$	$(1, 15, 86, 232, 289, 129, 0)$	$\{(-1, 0), (-3, 0), (0, 0)\}$
$(1, 2, 4)$	$(1, 16, 95, 261, 328, 147, 0)$	$\{(-1, 0), (-3, 0), (0, 0)\}$
$(1, 3, 4)$	$(1, 16, 96, 266, 336, 151, 0)$	$\{(-4, \pm 6), (-1, 0), (0, 0)\}$
$(2, 3, 4)$	$(1, 16, 96, 267, 338, 152, 0)$	$\{(-4, 0), (-1, 0), (-2, 0), (0, 0)\}$
$(0, 1, 2, 3)$	$(1, 15, 86, 232, 289, 130, 0)$	$\{(-2, 0), (0, 0)\}$
$(0, 1, 2, 4)$	$(1, 16, 95, 261, 328, 148, 0)$	$\{(-2, 0), (0, 0)\}$
$(0, 1, 3, 4)$	$(1, 16, 96, 266, 336, 152, 0)$	$\{(-2, 0), (0, 0)\}$
$(0, 2, 3, 4)$	$(1, 16, 96, 267, 338, 153, 0)$	$\{(0, 0)\}$
$(1, 2, 3, 4)$	$(1, 16, 96, 267, 339, 153, 0)$	$\{(-1, 0), (-3, 0), (0, 0)\}$
$(0, 1, 2, 3, 4)$	$(1, 16, 96, 267, 339, 154, 0)$	$\{(-2, 0), (0, 0)\}$

Table 2. Integer solutions of the Diophantine equation $y^2 = g_T(x)$ for $T \in A_5$.

Proof. In all cases $g_T(x)$ is a monic polynomial of degree 6, hence Runge's condition is satisfied. An algorithm to solve such Diophantine equations is given in [27], we followed it to determine the integral solutions. We provide details of the computation in case of $T = (2, 3)$, that is we deal with the Diophantine equation

$$y^2 = x^6 + 15\,x^5 + 86\,x^4 + 232\,x^3 + 288\,x^2 + 128\,x.$$

The polynomial part of the Puiseux expansion of $G_T(x)^{1/2}$ is given by

$$P_T(x) = x^3 + \frac{15}{2}\,x^2 + \frac{119}{8}\,x + \frac{71}{16}.$$

We have that

$$\begin{aligned} 256 g_T(x) - (16 P_T(x) - 1)^2 &= 32\,x^3 + 284\,x^2 - 552\,x - 4900, \\ 256 g_T(x) - (16 P_T(x) + 1)^2 &= -32\,x^3 - 196\,x^2 - 1504\,x - 5184. \end{aligned}$$

If $x > 4.154$, then

$$\begin{aligned} 256 g_T(x) - (16 P_T(x) - 1)^2 &> 0 \\ 256 g_T(x) - (16 P_T(x) + 1)^2 &< 0. \end{aligned}$$

Hence

$$(16 P_T(x) - 1)^2 < (16y)^2 < (16 P_T(x) + 1)^2.$$

It follows that $y = P_T(x)$. If $x < -8.875$, then

$$\begin{aligned} 256g_T(x) - (16P_T(x) - 1)^2 &< 0 \\ 256g_T(x) - (16P_T(x) + 1)^2 &> 0. \end{aligned}$$

Therefore we get that

$$(16P_T(x) + 1)^2 < (16y)^2 < (16P_T(x) - 1)^2.$$

We obtain that $y = P_T(x)$. If $x \in \{-8, \ldots, 4\}$, then the list of integral solutions (x, y) is given by $[(-4, 0), (-2, 0), (-1, 0), (0, 0), (2, \pm 72)]$. The equation $P_T(x)^2 - g_T(x) = 0$ does not possesses integral solutions. We note that the total running time to resolve the 31 equations was less than a second on an Intel Core i7-6700HQ 2.6GHz PC. □

Theorem 2.2. *For $T \in A_5$ consider the curve $C'_T : \ y^3 = g_T(x)$. The complete list of integral solutions are given Table 3.*

T	*polynomial*	$C'_T(\mathbb{Z})$
(0)	$(1, 15, 85, 225, 274, 121, 0)$	$\{(-1, -1), (0, 0)\}$
(1)	$(1, 15, 85, 225, 275, 121, 0)$	$\{(-1, 0), (0, 0)\}$
(2)	$(1, 15, 85, 226, 277, 122, 0)$	$\{(-1, 0), (-2, 0), (0, 0)\}$
(3)	$(1, 15, 86, 231, 285, 126, 0)$	$\{(-1, 0), (-2, 0), (-3, 0), (0, 0)\}$
(4)	$(1, 16, 95, 260, 324, 144, 0)$	$\{(-4, 0), (-1, 0), (-2, 0), (-6, 0), (-3, 0), (0, 0)\}$
$(0, 1)$	$(1, 15, 85, 225, 275, 122, 0)$	$\{(-1, -1), (-2, 0), (0, 0), (-4, 2)\}$
$(0, 2)$	$(1, 15, 85, 226, 277, 123, 0)$	$\{(-1, -1), (0, 0)\}$
$(0, 3)$	$(1, 15, 86, 231, 285, 127, 0)$	$\{(-1, -1), (0, 0)\}$
$(0, 4)$	$(1, 16, 95, 260, 324, 145, 0)$	$\{(-1, -1), (0, 0), (-5, -5)\}$
$(1, 2)$	$(1, 15, 85, 226, 278, 123, 0)$	$\{(-1, 0), (-3, 0), (0, 0)\}$
$(1, 3)$	$(1, 15, 86, 231, 286, 127, 0)$	$\{(-1, 0), (0, 0)\}$
$(1, 4)$	$(1, 16, 95, 260, 325, 145, 0)$	$\{(-1, 0), (0, 0)\}$
$(2, 3)$	$(1, 15, 86, 232, 288, 128, 0)$	$\{(-4, 0), (-1, 0), (-2, 0), (0, 0)\}$
$(2, 4)$	$(1, 16, 95, 261, 327, 146, 0)$	$\{(-1, 0), (-2, 0), (0, 0)\}$
$(3, 4)$	$(1, 16, 96, 266, 335, 150, 0)$	$\{(-1, 0), (-5, 0), (-2, 0), (-3, 0), (0, 0)\}$
$(0, 1, 2)$	$(1, 15, 85, 226, 278, 124, 0)$	$\{(-1, -1), (-2, 0), (1, 9), (0, 0)\}$
$(0, 1, 3)$	$(1, 15, 86, 231, 286, 128, 0)$	$\{(-1, -1), (-2, 0), (0, 0)\}$
$(0, 1, 4)$	$(1, 16, 95, 260, 325, 146, 0)$	$\{(-1, -1), (-2, 0), (0, 0), (-4, 2)\}$
$(0, 2, 3)$	$(1, 15, 86, 232, 288, 129, 0)$	$\{(-1, -1), (0, 0)\}$
$(0, 2, 4)$	$(1, 16, 95, 261, 327, 147, 0)$	$\{(-1, -1), (0, 0)\}$
$(0, 3, 4)$	$(1, 16, 96, 266, 335, 151, 0)$	$\{(-1, -1), (0, 0)\}$
$(1, 2, 3)$	$(1, 15, 86, 232, 289, 129, 0)$	$\{(-1, 0), (-3, 0), (0, 0)\}$
$(1, 2, 4)$	$(1, 16, 95, 261, 328, 147, 0)$	$\{(-1, 0), (-3, 0), (0, 0)\}$
$(1, 3, 4)$	$(1, 16, 96, 266, 336, 151, 0)$	$\{(-1, 0), (0, 0)\}$
$(2, 3, 4)$	$(1, 16, 96, 267, 338, 152, 0)$	$\{(-4, 0), (-1, 0), (-2, 0), (0, 0)\}$
$(0, 1, 2, 3)$	$(1, 15, 86, 232, 289, 130, 0)$	$\{(-1, -1), (-2, 0), (0, 0), (-4, 2)\}$
$(0, 1, 2, 4)$	$(1, 16, 95, 261, 328, 148, 0)$	$\{(-1, -1), (-2, 0), (0, 0)\}$
$(0, 1, 3, 4)$	$(1, 16, 96, 266, 336, 152, 0)$	$\{(-1, -1), (-2, 0), (0, 0)\}$
$(0, 2, 3, 4)$	$(1, 16, 96, 267, 338, 153, 0)$	$\{(-1, -1), (0, 0)\}$
$(1, 2, 3, 4)$	$(1, 16, 96, 267, 339, 153, 0)$	$\{(-1, 0), (-3, 0), (0, 0)\}$
$(0, 1, 2, 3, 4)$	$(1, 16, 96, 267, 339, 154, 0)$	$\{(-1, -1), (-2, 0), (0, 0), (-4, 2)\}$

Table 5. Integer solutions of the Diophantine equation $y^3 = g_T(x)$ for $T \in A_5$.

Proof. Here we provide details of the computation in case of $T = (0, 4)$, that is we consider the Diophantine equation

$$y^3 = x^6 + 16\,x^5 + 95\,x^4 + 260\,x^3 + 324\,x^2 + 145\,x.$$

The polynomial part of the Puiseux expansion of $g_T(x)^{1/3}$ is given by

$$P_T(x) = x^2 + \frac{16}{3}\,x + \frac{29}{9}.$$

We obtain that

$$\begin{aligned} 729g_T(x) - (9P_T(x) - 1)^3 &= 243\,x^4 + 6372\,x^3 + 21492\,x^2 - 7191\,x - 21952, \\ 729g_T(x) - (9P_T(x) + 1)^3 &= -243\,x^4 + 1188\,x^3 + 4536\,x^2 - 23895\,x - 27000. \end{aligned}$$

If $x > 1.019$ or $x < -22.184$, then

$$\begin{aligned} 729g_T(x) - (9P_T(x) - 1)^3 &> 0 \\ 729g_T(x) - (9P_T(x) + 1)^3 &< 0. \end{aligned}$$

Thus we get that

$$(9P_T(x) - 1)^3 < (9y)^3 < (9P_T(x) + 1)^3.$$

It follows that $y = P_T(x)$. If $-22 \leq x \leq 1$, then the list of integral solutions is given by $[(-5, -5), (-1, -1), (0, 0)]$. The equation $g_T(x) - P_T(x)^3 = 0$ has no integral solution. The total running time to resolve the 31 equations corresponding to the case $(m, n) = (3, 5)$ was less than a second on an Intel Core i7-6700HQ 2.6GHz PC. □

We also computed the integral solutions of the equations $y^2 = g_T(x)$ with $T \in A_7, A_9, A_{11}$ and A_{13}. The total running times on an Intel Core i7-6700HQ 2.6GHz PC are given below.

n	running time
7	14s
9	2 min 48s
11	1h 39min 53s
13	27h 12min 38s

Let us collect some data related to solutions with $xy \neq 0$. If $T \in A_7$, then $(-4, \pm 6)$ is a solution if T contains 1,3 and some of the numbers 4, 5, 6. Other solutions are given by $(1, \pm 213)$ if $T = (0, 1, 2, 6)$ and $(1, \pm 215)$ if $T = (0, 3, 4, 5, 6)$. If $T \in A_9$, then $(-4, \pm 6)$ is a solution if T contains 1,3 and some of the numbers 4, 5, 6, 7, 8. The remaining solutions are given by

T	solutions
$(3, 5)$	$[(-9, \pm 252)]$
$(8, 3, 5, 7)$	$[(-9, \pm 252)]$
$(0, 1, 2, 5, 6, 7)$	$[(1, \pm 1917)]$

If $T \in A_{11}$, then $(-4, \pm 6)$ is a solution if T contains 1,3 and some of the numbers 4, 5, 6, 7, 8, 9, 10. The following table contains the rest of the solutions

T	solutions
$(3, 5)$	$[(-9, \pm 252)]$
$(9, 3, 5)$	$[(-9, \pm 252)]$
$(10, 3, 5)$	$[(-9, \pm 252)]$
$(8, 3, 5, 7)$	$[(-9, \pm 252)]$
$(9, 10, 3, 5)$	$[(-9, \pm 252)]$
$(8, 9, 3, 5, 7)$	$[(-9, \pm 252)]$
$(8, 10, 3, 5, 7)$	$[(-9, \pm 252)]$
$(3, 5, 7, 8, 9, 10)$	$[(-9, \pm 252)]$

If $T \in A_{13}$, then $(-4, \pm 6)$ is a solution if T contains 1,3 and some of the numbers 4, 5, 6, 7, 8, 9, 10, 11, 12. If T is an element of the following list, then $(-9, \pm 252)$ is a solution:

$$(3, 5), (9, 3, 5), (10, 3, 5), (3, 11, 5), (3, 12, 5), (8, 3, 5, 7), (9, 10, 3, 5), (3, 9, 11, 5), (9, 3, 12, 5), (3, 10, 11, 5),$$
$$(10, 3, 12, 5), (3, 11, 12, 5), (8, 9, 3, 5, 7), (8, 10, 3, 5, 7), (8, 3, 11, 5, 7), (8, 3, 12, 5, 7), (3, 9, 10, 11, 5),$$
$$(9, 10, 3, 12, 5), (3, 9, 11, 12, 5), (3, 10, 11, 12, 5), (3, 5, 7, 8, 9, 10), (3, 5, 7, 8, 9, 11), (3, 5, 7, 8, 9, 12),$$
$$(3, 5, 7, 8, 10, 11), (3, 5, 7, 8, 10, 12), (3, 5, 7, 8, 11, 12), (3, 5, 9, 10, 11, 12), (3, 5, 7, 8, 9, 10, 11), (3, 5, 7, 8, 9, 10, 12),$$
$$(3, 5, 7, 8, 9, 11, 12), (3, 5, 7, 8, 10, 11, 12), (3, 5, 7, 8, 9, 10, 11, 12).$$

3. Some results concerning additive version of Erdős-Graham question

Theorem 3.1. *There are infinitely many integer solutions of the Diophantine equation*

$$z^m = p_1(x) + p_1(y) \tag{2}$$

for m odd and $m = 2, 4$.

Proof. If $m \equiv 1 \pmod 2$ then the triple

$$x = 2^{\frac{n-1}{2}} t^m - 1, \quad y = 2^{\frac{n-1}{2}} t^m, \quad z = 2t^2$$

is the solution of the equation (2).

In case of $m = 3$ we offer a solution which do not satisfy the condition $y - x = 1$, i.e.,

$$\begin{aligned} x &= 170t^3 - 684t^2 + 906t - 395, \\ y &= 3(34t^3 - 164t^2 + 258t - 133), \\ z &= 2(17t^2 - 48t + 34). \end{aligned}$$

In order to get the above parametrization, we first performed computer search for solutions in the range $0 < x < y < 10^5$ with the assumption that $y - x \neq 1$. Then, after a careful inspection of the solutions set we found the presented family.

Now we consider the case $m = 2$. We observe that for any given positive integer k the triple of polynomials

$$\begin{aligned} x &= 4kt^2 + (4k-1)t - k - 1, \\ y &= (2k-1)(2k+1)t^2 + (4k^2 - 2k - 1)t - k(k+1), \\ z &= (4k^2+1)t^2 + (4k^2 - 2k + 1)t - k(k+1). \end{aligned}$$

solves the equation $z^2 = p_1(x) + p_1(y)$.

We use the above solution in order to prove that the equation (2), with $m = 4$, has infinitely many solutions in integers. In order to do this, it is enough to prove the existence of $k_0 \in \mathbb{N}$, such that the quadratic equation

$$v^2 = (4k_0^2 + 1)t^2 + (4k_0^2 - 2k_0 + 1)t - k_0(k_0 + 1)$$

has infinitely many solutions. By taking $k_0 = 1$ we get the Pell type equation

$$v^2 = 5t^2 + 3t - 2.$$

Using the standard technique we get the formula for t as follows

$$t = \frac{7(9 + 4\sqrt{5})^{n+1} + 7(9 - 4\sqrt{5})^{n+1} - 6}{20},$$

for $n \in \mathbb{N}_+$. However, one can check that $t = t_n \in \mathbb{Z}$ if and only if n is even. Thus, taking even values of n we get the solutions we are looking for. As an example, let us note the first three smallest solutions of the equation $z^4 = p_1(x) + p_1(y)$ obtained by our approach. They are given by

$$(x, y) = (2022, 4522), (651174, 1456070), (209676102, 468850018).$$

□

Remark 3.2. It is clear that with the method used to get infinitely many solutions of the equation $z^4 = p_1(x) + p_1(y)$ one can construct infinitely many families generated by appropriate Pell type equations. Indeed, let us write $Q(k, t) = (4k^2 + 1)t^2 + (4k^2 - 2k + 1)t - k(k + 1)$. We have proved that the equation $z^2 = Q(k_0, t)$ has infinitely many integer solutions $t = t_{0,n} := t_{2n}$. Now, for each $n \in \mathbb{N}$ we can treat the equation $z^2 = Q(k, t_{0,n})$ as Pell type equation in variables (k, z) with known solution $(k, z) = (k_0, z_{0,n})$, where $z_{0,n} = \sqrt{Q(k_0, t_{0,n})}$. In this way, for each $n \in \mathbb{N}$ we can construct an infinite sequence $(k_{m,n})_{m \in \mathbb{N}}$ such that $Q(k_{m,n}, t_{0,n})$ is a square for each $m, n \in \mathbb{N}$. Thus, we get infinitely many families of integer solutions of the equation $z^4 = p_1(x) + p_1(t)$.

Consider the Diophantine equation

$$z^2 = p_2(x) + p_2(y). \tag{3}$$

Theorem 3.3. *The Diophantine equation* (3) *has infinitely many solutions* (x, y, z) *in positive integers.*

Proof. We have that

$$p_2(x) + p_2(y) = (x + y + 2)(x^2 - xy + y^2 + x + y). \tag{4}$$

We show that there are infinitely many positive integer solutions of the system of equations

$$\begin{aligned} x + y + 2 &= z_1^2 \\ x^2 - xy + y^2 + x + y &= z_2^2. \end{aligned}$$

From the second equation we obtain an infinite family of solutions given by

$$x = 3t^2 + 4t - 1, \quad y = 2t, \quad z_2 = 3t^2 + 3t.$$

It remains to determine possible values of t for which $x + y + 2$ is a square. We have that

$$T^2 - 3(t+1)^2 = -2,$$

a Pell type equation. From the theory of Pell equations we get the formula for t as follows

$$t = \frac{(1+\sqrt{3})(2+\sqrt{3})^n - (1-\sqrt{3})(2-\sqrt{3})^n}{2\sqrt{3}} - 1, \quad n \in \mathbb{N}.$$

The first few solutions are given by

$$(x, y) = (19, 4), (339, 20), (4959, 80), (69919, 304), (976979, 1140).$$

□

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