NOTE ON A PAPER "AN EXTENSION OF A THEOREM OF EULER" BY HIRATA-KOHNO ET AL.

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ABSTRACT. In this paper we extend a result of Hirata-Kohno, Laishram, Shorey and Tijdeman on the Diophantine equation $n(n+d)\cdots(n+(k-1)d) = by^2$, where $n, d, k \geq 2$ and y are positive integers such that $\gcd(n, d) = 1$.

1. INTRODUCTION

Let $n, d, k > 2$ and y be positive integers such that $\gcd(n, d) = 1$. For an integer $\nu > 1$, we denote by $P(\nu)$ the greatest prime factor of ν and we put $P(1) = 1$. Let b be a squarefree positive integer such that $P(b) \leq k$. We consider the equation

$$n(n+d)\cdots(n+(k-1)d) = by^2 \tag{1}$$

in n, d, k and y.

A celebrated theorem of Erdős and Selfridge [7] states that the product of consecutive positive integers is never a perfect power. An old, difficult conjecture states that even a product of consecutive terms of arithmetic progression of length $k > 3$ and difference $d \geq 1$ is never a perfect power. Euler proved (see [6] pp. 440 and 635) that a product of four terms in arithmetic progression is never a square solving equation (1) with $b = 1$ and $k = 4$. Obláth [10] obtained a similar statement for $b = 1, k = 5$. Bennett, Bruin, Győry and Hajdu [1] solved (1) with $b = 1$ and $6 \leq k \leq 11$. For more results on this topic see [1], [8] and the references given there.

We write

$$n + id = a_i x_i^2 \text{ for } 0 \leq i < k \tag{2}$$

where a_i are squarefree integers such that $P(a_i) \leq \max(P(b), k-1)$ and x_i are positive integers. Every solution to (1) yields a k-tuple $(a_0, a_1, \ldots, a_{k-1})$. Recently Hirata-Kohno, Laishram, Shorey and Tijdeman [8] proved the following theorem.

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Theorem A (Hirata-Kohno, Laishram, Shorey,Tijdeman)**.** *Equation* (1) *with* $d > 1, P(b) = k$ *and* $7 \leq k \leq 100$ *implies that* $(a_0, a_1, \ldots, a_{k-1})$ *is among the following tuples or their mirror images.*

$$\begin{aligned} k = 7: \quad & (2,3,1,5,6,7,2),(3,1,5,6,7,2,1),(1,5,6,7,2,1,10), \\ k = 13: \quad & (3,1,5,6,7,2,1,10,11,3,13,14,15), \\ & (1,5,6,7,2,1,10,11,3,13,14,15,1), \\ k = 19: \quad & (1,5,6,7,2,1,10,11,3,13,14,15,1,17,2,19,5,21,22), \\ k = 23: \quad & (5,6,7,2,1,10,11,3,13,14,15,1,17,2,19,5,21,22,23,6,1,26,3), \\ & (6,7,2,1,10,11,3,13,14,15,1,17,2,19,5,21,22,23,6,1,26,3,7). \end{aligned}$$

In case of $k = 5$ Mukhopadhyay and Shorey [9] proved the following result.

Theorem B (Mukhopadhyay, Shorey)**.** *If* n *and* d *are coprime nonzero integers, then the Diophantine equation*

$$n(n+d)(n+2d)(n+3d)(n+4d) = by^2$$

has no solutions in nonzero integers b, y *and* $P(b) \leq 3$.

In this article we solve (1) with $k = 5$ and $P(b) = 5$, moreover we handle the 8 special cases mentioned in Theorem A. We prove the following theorems.

Theorem 1. *Equation* (1) *with* $d > 1, P(b) = k$ *and* $7 \leq k \leq 100$ *has no solutions.*

Theorem 2. *Equation* (1) *with* $d > 1, k = 5$ *and* $P(b) = 5$ *implies that* $(n, d) \in \{(-12, 7), (-4, 3)\}$.

2. Preliminary lemmas

In the proofs of Theorem 2 and 1 we need several results using elliptic Chabauty's method (see [4],[5]). Bruin's routines related to elliptic Chabauty's method are contained in MAGMA [2]. Here we only indicate the main steps without explaining the background theory. To see how the method works in practice, in particular by the help of Magma, [3] is an excellent source. To have the method work, the rank of the elliptic curve (defined over the number field K) should be strictly less than the degree of K. In the present cases it turns out that the ranks of the elliptic curves are either 0 or 1, so elliptic Chabauty's method is applicable. Further, the procedure `PseudoMordellWeilGroup` of Magma is able to find a subgroup of the Mordell-Weil group of finite odd index. We also need to check that the index is not divisible by some prime numbers provided by the procedure `Chabauty`. This last step can be done by the inbuilt function `IsPSaturated`.

Lemma 1. *Equation* (1) *with* $k=7$ *and* $(a_0,a_1,\dots,a_6)=(1,5,6,7,2,1,10)$ *implies that* $n=2, d=1$.

Proof. Using that $n=x_0^2$ and $d=(x_5^2-x_0^2)/5$ we obtain the following system of equations

$$\begin{aligned} x_5^2+4x_0^2&=25x_1^2,\\ 4x_5^2+x_0^2&=10x_4^2,\\ 6x_5^2-x_0^2&=50x_6^2. \end{aligned}$$

The second equation implies that x_0 is even, that is there exists a $z\in\mathbb{Z}$ such that $x_0=2z$. By standard factorization argument in the Gaussian integers we get that

$$(x_5+4iz)(x_5+iz)=\delta\square,$$

where $\delta\in\{-3\pm i,-1\pm 3i,1\pm 3i,3\pm i\}$. Thus putting $X=x_5/z$ it is sufficient to find all points (X,Y) on the curves

$$C_\delta:\quad \delta(X+i)(X+4i)(3X^2-2)=Y^2, \tag{3}$$

where $\delta\in\{-3\pm i,-1\pm 3i,1\pm 3i,3\pm i\}$, for which $X\in\mathbb{Q}$ and $Y\in\mathbb{Q}(i)$. Note that if (X,Y) is a point on C_δ then (X,iY) is a point on $C_{-\delta}$. We will use this isomorphism later on to reduce the number of curves to be examined. Hence we need to consider the curve C_δ for $\delta\in\{1-3i,1+3i,3-i,3+i\}$.

I. $\delta=1-3i$. In this case C_{1-3i} is isomorphic to the elliptic curve

$$E_{1-3i}:\quad y^2=x^3+ix^2+(-17i-23)x+(2291i+1597).$$

Using MAGMA we get that the rank of E_{1-3i} is 0 and there is no point on C_{1-3i} for which $X\in\mathbb{Q}$.

II. $\delta=1+3i$. Here we obtain that $E_{1+3i}: y^2=x^3-ix^2+(17i-23)x+(-2291i+1597)$. The rank of this curve is 0 and there is no point on C_{1+3i} for which $X\in\mathbb{Q}$.

III. $\delta=3-i$. The elliptic curve in this case is $E_{3-i}: y^2=x^3+x^2+(-17i+23)x+(-1597i-2291)$. We have $E_{3-i}(\mathbb{Q}(i))\simeq\mathbb{Z}_2\oplus\mathbb{Z}$ as an Abelian group. Applying elliptic Chabauty with $p=13$, we get that $x_5/z=-3$. Thus $n=2$ and $d=1$.

IV. $\delta=3+i$. The curve C_{3+i} is isomorphic to $E_{3+i}: y^2=x^3+x^2+(17i+23)x+(1597i-2291)$. The rank of this curve is 1 and applying elliptic Chabauty again with $p=13$ we obtain that $x_5/z=3$. This implies that $n=2$ and $d=1$. $\square$

Lemma 2. *Equation* (1) *with* $k=7$ *and* $(a_0,a_1,\dots,a_6)=(2,3,1,5,6,7,2)$ *implies that* $n=2, d=1$.

Proof. In this case we have the following system of equations

$$x_4^2 + x_0^2 = 2x_1^2,$$
$$9x_4^2 + x_0^2 = 10x_3^2,$$
$$9x_4^2 - x_0^2 = 2x_6^2.$$

Using the same argument as in the proof of Theorem 1 it follows that it is sufficient to find all points (X, Y) on the curves

$$C_\delta : \quad 2\delta(X+i)(3X+i)(9X^2-1) = Y^2, \tag{4}$$

where $\delta \in \{-4 \pm 2i, -2 \pm 4i, 2 \pm 4i, 4 \pm 2i\}$, for which $X \in \mathbb{Q}$ and $Y \in \mathbb{Q}(i)$. We summarize the results obtained by elliptic Chabauty in the following table. In each case we used $p = 29$.

δ	curve	x_4/x_0
$2-4i$	$y^2 = x^3 + (-12i-9)x + (-572i-104)$	$\{-1, \pm 1/3\}$
$2+4i$	$y^2 = x^3 + (12i-9)x + (-572i+104)$	$\{1, \pm 1/3\}$
$4-2i$	$y^2 = x^3 + (-12i+9)x + (-104i-572)$	$\{\pm 1/3\}$
$4+2i$	$y^2 = x^3 + (12i+9)x + (-104i+572)$	$\{\pm 1/3\}$

Thus $x_4/x_0 \in \{\pm 1, \pm 1/3\}$. From $x_4/x_0 = \pm 1$ it follows that $n = 2, d = 1$, while $x_4/x_0 = \pm 1/3$ does not yield any solutions. □

Lemma 3. *Equation* (1) *with* $k = 7$ *and* $(a_0, a_1, \ldots, a_6) = (3, 1, 5, 6, 7, 2, 1)$ *implies that* $n = 3, d = 1$.

Proof. Here we get the following system of equations

$$2x_3^2 + 2x_0^2 = x_1^2,$$
$$4x_3^2 + x_0^2 = 5x_2^2,$$
$$12x_3^2 - 3x_0^2 = x_6^2.$$

Using the same argument as in the proof of Theorem 1 it follows that it is sufficient to find all points (X, Y) on the curves

$$C_\delta : \quad \delta(X+i)(2X+i)(12X^2-3) = Y^2, \tag{5}$$

where $\delta \in \{-3 \pm i, -1 \pm 3i, 1 \pm 3i, 3 \pm i\}$ for which $X \in \mathbb{Q}$ and $Y \in \mathbb{Q}(i)$. We summarize the results obtained by elliptic Chabauty in the following table. In each case we used $p = 13$.

δ	curve	x_3/x_0
$1-3i$	$y^2 = x^3 + (27i+36)x + (243i-351)$	$\{-1, \pm 1/2\}$
$1+3i$	$y^2 = x^3 + (-27i+36)x + (243i+351)$	$\{1, \pm 1/2\}$
$3-i$	$y^2 = x^3 + (27i-36)x + (-351i+243)$	$\{\pm 1/2\}$
$3+i$	$y^2 = x^3 + (-27i-36)x + (-351i-243)$	$\{\pm 1/2\}$

Thus $x_3/x_0 \in \{\pm 1, \pm 1/2\}$. From $x_4/x_0 = \pm 1$ it follows that $n = 3, d = 1$, while $x_3/x_0 = \pm 1/2$ does not yield any solutions. □

Lemma 4. *Equation* (1) *with* $(a_0, a_1, \ldots, a_4) = (-3, -5, 2, 1, 1)$ *and* $k = 5, d > 1$ *implies that* $n = -12, d = 7$.

Proof. From the system of equations (2) we have

$$\begin{aligned} \frac{1}{4}x_4^2 - \frac{9}{4}x_0^2 &= -5x_1^2, \\ \frac{1}{2}x_4^2 - \frac{3}{2}x_0^2 &= 2x_2^2, \\ \frac{3}{4}x_4^2 - \frac{3}{4}x_0^2 &= x_3^2. \end{aligned}$$

Clearly, $\gcd(x_4, x_0) = 1$ or 2. In both cases we get the following system of equations

$$\begin{aligned} X_4^2 - 9X_0^2 &= -5\square, \\ X_4^2 - 3X_0^2 &= \square, \\ X_4^2 - X_0^2 &= 3\square, \end{aligned}$$

where $X_4 = x_4/\gcd(x_4, x_0)$ and $X_0 = x_0/\gcd(x_4, x_0)$. The curve in this case is

$$C_\delta: \quad \delta(X + \sqrt{3})(X + 3)(X^2 - 1) = Y^2,$$

where δ is from a finite set. Elliptic Chabauty's method applied with $p = 11, 37$ and 59 provides all points for which the first coordinate is rational. These coordinates are $\{-3, -2, -1, 1, 2\}$. We obtain the arithmetic progression with $(n, d) = (-12, 7)$. $\square$

Lemma 5. *Equation* (1) *with* $(a_0, a_1, \ldots, a_4) = (2, 5, 2, -1, -1)$ *and* $k = 5, d > 1$ *implies that* $n = -4, d = 3$.

Proof. We use x_3 and x_2 to get a system of equations as in the previous lemmas. Elliptic Chabauty's method applied with $p = 13$ yields that $x_3/x_2 = \pm 1$, hence $(n, d) = (-4, 3)$. $\square$

Lemma 6. *Equation* (1) *with* $(a_0, a_1, \ldots, a_4) = (6, 5, 1, 3, 2)$ *and* $k = 5, d > 1$ *has no solutions.*

Proof. In this case we have

$$\delta(x_3 + \sqrt{-1}x_0)(x_3 + 2\sqrt{-1}x_0)(2x_3^2 - x_0^2) = \square,$$

where $\delta \in \{1 \pm 3\sqrt{-1}, 3 \pm \sqrt{-1}\}$. Chabauty's argument gives $x_3/x_0 = \pm 1$, which corresponds to arithmetic progressions with $d = \pm 1$. $\square$

3. Remaining cases of Theorem A

In this section we prove Theorem 1.

Proof. First note that Lemmas 1, 2 and 3 imply the statement of the theorem in cases of $k = 7, 13$ and 19. The two remaining possibilities can be eliminated in a similar way, we present the argument working for the tuple

$$(5, 6, 7, 2, 1, 10, 11, 3, 13, 14, 15, 1, 17, 2, 19, 5, 21, 22, 23, 6, 1, 26, 3).$$

We have the system of equations

$$\begin{aligned} n + d &= 6x_1^2, \\ n + 3d &= 2x_3^2, \\ n + 5d &= 10x_5^2, \\ n + 7d &= 3x_7^2, \\ n + 9d &= 14x_9^2, \\ n + 11d &= x_{11}^2, \\ n + 13d &= 2x_{13}^2. \end{aligned}$$

We find that x_7, x_{11} and $(n + d)$ are even integers. Dividing all equations by 2 we obtain an arithmetic progression of length 7 and $(a_0, a_1, \ldots, a_6) = (3, 1, 5, 6, 7, 2, 1)$. This is not possible by Lemma 3 and the theorem is proved. □

4. THE CASE $k = 5$

In this section we prove Theorem 2.

Proof. Five divides one of the terms and by symmetry we may assume that $5 \mid n + d$ or $5 \mid n + 2d$. First we compute the set of possible tuples $(a_0, a_1, a_2, a_3, a_4)$ for which appropriate congruence conditions hold ($\gcd(a_i, a_j) \in \{1, P(j - i)\}$ for $0 \leq i < j \leq 4$) and the number of sign changes are at most 1 and the product $a_0a_1a_2a_3a_4$ is positive. After that we eliminate tuples by using elliptic curves of rank 0. We consider elliptic curves $(n + \alpha_1 d)(n + \alpha_2 d)(n + \alpha_3 d)(n + \alpha_4 d) = \prod_i a_{\alpha_i} \square$, where $\alpha_i, i \in \{1, 2, 3, 4\}$ are distinct integers belonging to the set $\{0, 1, 2, 3, 4\}$. If the rank is 0, then we obtain all possible values of n/d. Since $\gcd(n, d) = 1$ we get all possible values of n and d. It turns out that it remains to deal with the following tuples

$$\begin{aligned} &(-3, -5, 2, 1, 1), \\ &(-2, -5, 3, 1, 1), \\ &(-1, -15, -1, -2, 3), \\ &(2, 5, 2, -1, -1), \\ &(6, 5, 1, 3, 2). \end{aligned}$$

In case of $(-3, -5, 2, 1, 1)$ Lemma 4 implies that $(n, d) = (-12, 7)$.

If $(a_0, a_1, \ldots, a_4) = (-2, -5, 3, 1, 1)$, then by $\gcd(n, d) = 1$ we have that $\gcd(n, 3) = 1$. Since $n = -2x_0^2$ we obtain that $n \equiv 1 \pmod 3$. From the equation $n + 2d = 3x_2^2$ we get that $d \equiv 1 \pmod 3$. Finally, the equation $n + 4d = x_4^2$ leads to a contradiction.

If $(a_0, a_1, \ldots, a_4) = (-1, -15, -1, -2, 3)$, then we obtain that $\gcd(n, 3) = 1$. From the equations $n = -x_0^2$ and $n + d = -15x_1^2$ we get that $n \equiv 2 \pmod 3$ and $d \equiv 1 \pmod 3$. Now the contradiction follows from the equation $n + 2d = -x_2^2$.

In case of the tuple $(2, 5, 2, -1, -1)$ Lemma 5 implies that $(n, d) = (-4, 3)$. The last tuple is eliminated by Lemma 6. □

References

[1] M. A. Bennett, N. Bruin, K. Győry, and L. Hajdu. Powers from products of consecutive terms in arithmetic progression. *Proc. London Math. Soc. (3)*, 92(2):273–306, 2006.

[2] W. Bosma, J. Cannon, and C. Playoust. The Magma algebra system. I. The user language. *J. Symbolic Comput.*, 24(3-4):235–265, 1997. Computational algebra and number theory (London, 1993).

[3] N. Bruin. Some ternary diophantine equations of signature $(n, n, 2)$. In Wieb Bosma and John Cannon, editors, *Discovering Mathematics with Magma — Reducing the Abstract to the Concrete*, volume 19 of *Algorithms and Computation in Mathematics*, pages 63–91. Springer, Heidelberg, 2006.

[4] N. R. Bruin. *Chabauty methods and covering techniques applied to generalized Fermat equations*, volume 133 of *CWI Tract*. Stichting Mathematisch Centrum Centrum voor Wiskunde en Informatica, Amsterdam, 2002. Dissertation, University of Leiden, Leiden, 1999.

[5] Nils Bruin. Chabauty methods using elliptic curves. *J. Reine Angew. Math.*, 562:27–49, 2003.

[6] L.E. Dickson. *History of the theory of numbers. Vol II: Diophantine analysis.* Chelsea Publishing Co., New York, 1966.

[7] P. Erdős and J. L. Selfridge. The product of consecutive integers is never a power. *Illinois J. Math.*, 19:292–301, 1975.

[8] N. Hirata-Kohno, S. Laishram, T.N. Shorey, and R. Tijdeman. An Extension of a Theorem of Euler. *preprint.*

[9] Anirban Mukhopadhyay and T. N. Shorey. Almost squares in arithmetic progression. II. *Acta Arith.*, 110(1):1–14, 2003.

[10] Richard Obláth. Über das Produkt fünf aufeinander folgender Zahlen in einer arithmetischen Reihe. *Publ. Math. Debrecen*, 1:222–226, 1950.

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