Diophantine Problems and Arithmetic Progressions

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Background

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- Obláth obtained a similar statement for $k = 5$.

Background

- Many nice results by Bruin, Bennett, Győry, Hajdu, Laishram, Pintér, Saradha, Shorey and others related to the Diophantine equation
- $f(x, k, d) = by^l$
- Techniques: Baker's method, modular approach, theory of elliptic curves, Chabauty's method, high degree Thue equations.

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Erdős and Graham asked if the Diophantine equation

$$\prod_{i=1}^{r} f(x_i, k_i, 1) = y^2$$

has, for fixed $r \geq 1$ and $\{k_1, k_2, \ldots, k_r\}$ with $k_i \geq 4$ for $i = 1, 2, \ldots, r$, at most finitely many solutions in positive integers $(x_1, x_2, \ldots, x_r, y)$ with $x_i + k_i \leq x_{i+1}$ for $1 \leq i \leq r-1$.

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- Bauer and Bennett (2007) extended this result to the cases $r = 3$ and $r = 5$.
- Luca and Walsh (2007) studied the case $(r, k_i) = (2, 4)$.
- Bennett and Van Luijk (2012) constructed an infinite family of $r \geq 5$ non-overlapping blocks of five consecutive integers such that their product is always a perfect square.

Product of two blocks

We deal with the Diophantine equation

$$x(x+1)(x+2)(x+3)(x+k)(x+k+1)(x+k+2)(x+k+3) = y^2. \quad (1)$$

Theorem (Sz. T. (2015))

If $(x, y) \in \mathbb{N}^2$ *is a solution of* (1) *then*

$$1 \leq x \leq 1.08k.$$

Theorem (Sz. T. (2015))

The only solution $(x, y) \in \mathbb{N}^2$ *of* (1) *with* $4 \leq k \leq 10^6$ *is*

$$(x, y) = (33, 3361826160)$$

with $k = 1647$.

Proof of the Theorems

We apply Runge's method and we prove that large solutions do not exists and we provide bound for size of the possible small solutions. A solution to equation (1) gives rise a solution to the equation

$$F(X) := X(X+k+2)(X+2k+2)(X+3k) = Y^2, \qquad (2)$$

where $X = x^2 + (k+3)x$. The polynomial part of the Puiseux expansion of $F(X)^{(1/2)}$ is

$$P(X) = X^2 + (3k+2)X + k^2 + 3k.$$

Proof of the Theorems

We obtain that

$$\begin{aligned} F(X)-(P(X)-1)^2 &= 2X^2-(4k^2-6k+4)X-k^4-6k^3-7k^2+6k-1, \\ F(X)-(P(X)+1)^2 &= -2X^2-(4k^2+6k+4)X-k^4-6k^3-11k^2-6k-1. \end{aligned}$$

Let α_1, α_2 be the roots of the quadratic polynomial $F(X)-(P(X)-1)^2$ and α_3, α_4 be the roots of $F(X)-(P(X)+1)^2$. We define $\beta_i, i = 1, 2, 3, 4$ as follows

$$\beta_i = \begin{cases} \alpha_i & \text{if } \alpha_i \in \mathbb{R}, \\ 0 & \text{otherwise.} \end{cases}$$

Proof of the Theorems

It follows that

$$F(X) - (P(X) - 1)^2 > 0, \quad \text{if } X \notin [\min_i\{\beta_i\}, \max_i\{\beta_i\}]$$

and

$$F(X) - (P(X) + 1)^2 < 0, \quad \text{if } X \notin [\min_i\{\beta_i\}, \max_i\{\beta_i\}].$$

Hence we get that

$$(P(X) - 1)^2 < F(X) < (P(X) + 1)^2, \quad \text{if } X \notin [\min_i\{\beta_i\}, \max_i\{\beta_i\}].$$

If (X, Y) is a solution of (2) with $X \notin [\min_i\{\beta_i\}, \max_i\{\beta_i\}]$, then

$$Y = P(X).$$

It implies that

$$0 = F(X) - P(X)^2 = -4k^2X - k^4 - 6k^3 - 9k^2.$$

That is

$$X = -\left(\frac{k+3}{2}\right)^2.$$

Large solutions

Since $X = x^2 + (k+3)x$ we get that

$$x = \frac{-k-3}{2}.$$

It means that if there exists a large solution, then k has to be odd, $x = \frac{-k-3}{2}$ and $y = \frac{(k-3)(k-1)(k+1)(k+3)}{16}$. It is a contradiction since $k \geq 4$ and therefore $0 > \frac{-k-3}{2} = x$.

Small solutions

$$\begin{aligned}\alpha_1 &= k^2 - \frac{3}{2}\,k - 1 - \frac{1}{2}\,\sqrt{6\,k^4 + 15\,k^2 + 6},\\ \alpha_2 &= k^2 - \frac{3}{2}\,k - 1 + \frac{1}{2}\,\sqrt{6\,k^4 + 15\,k^2 + 6},\\ \alpha_3 &= -k^2 - \frac{3}{2}\,k - 1 - \frac{1}{2}\,\sqrt{2\,k^4 - 5\,k^2 + 2},\\ \alpha_4 &= -k^2 - \frac{3}{2}\,k - 1 + \frac{1}{2}\,\sqrt{2\,k^4 - 5\,k^2 + 2}.\end{aligned}$$

Since $k \geq 4$, we obtain that $6\,k^4 + 15\,k^2 + 6 \geq 0$ and $2\,k^4 - 5\,k^2 + 2 \geq 0$. Therefore $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbb{R}$ and we have

$$\alpha_3 < \alpha_4 < \alpha_1 < \alpha_2.$$

Small solutions

We need to solve the system of inequalities

$$\begin{aligned} 0 &\leq x^2 + (k+3)x - \alpha_3, \\ 0 &\geq x^2 + (k+3)x - \alpha_2. \end{aligned}$$

The first inequality is true for all $x \geq 1$. The second inequality implies that

$$-\frac{1}{2}\,k - \frac{1}{2}\,\sqrt{5\,k^2 + 2\,\sqrt{6\,k^4 + 15\,k^2 + 6} + 5} - \frac{3}{2} \leq x$$

and

$$x \leq -\frac{1}{2}\,k + \frac{1}{2}\,\sqrt{5\,k^2 + 2\,\sqrt{6\,k^4 + 15\,k^2 + 6} + 5} - \frac{3}{2}.$$

The lower bound is negative if $k > 0$, hence we have that $x > 0$, in case of the upper bound we obtain that $x \leq 1.08k$ if $k \geq 4$.

Elliptic curve

$$X(X+k+2)(X+2k+2)(X+3k) = Y^2$$

k	solutions $(X, Y) \in \mathbb{Z}^2$
4	$(-15, \pm 45), (-12, 0), (-10, 0), (-9, \pm 9), (-6, 0), (0, 0), (6, \pm 144)$
5	$(-21, \pm 126), (-16, \pm 24), (-15, 0), (-12, 0), (-9, \pm 18), (-7, 0), (3, \pm 90), (0, 0)$
6	$(-28, \pm 280), (-18, 0), (-14, 0), (-12, \pm 24), (-8, 0), (2, \pm 80), (0, 0)$
7	$(-36, \pm 540), (-25, \pm 120), (-21, 0), (-16, 0), (-12, \pm 36), (-9, 0), (0, 0)$
8	$(-45, \pm 945), (-24, 0), (-18, 0), (-15, \pm 45), (-10, 0), (0, 0)$
9	$(-55, \pm 1540), (-36, \pm 360), (-27, 0), (-20, 0), (-15, \pm 60), (-11, 0), (0, 0), (1, \pm 84)$
10	$(-66, \pm 2376), (-30, 0), (-22, 0), (-18, \pm 72), (-12, 0), (0, 0)$
11	$(-78, \pm 3510), (-49, \pm 840), (-33, 0), (-24, 0), (-22, \pm 66), (-18, \pm 90), (-13, 0), (0, 0), (3, \pm 216)$
12	$(-91, \pm 5005), (-42, \pm 336), (-39, \pm 195), (-36, 0), (-26, 0), (-25, \pm 55), (-21, \pm 105)$ $(-16, \pm 80), (-14, 0), (0, 0), (13, \pm 819), (28, -2016)$
13	$(-105, \pm 6930), (-64, \pm 1680), (-39, 0), (-28, 0), (-21, \pm 126), (-15, 0), (0, 0)$
14	$(-120, \pm 9360), (-42, 0), (-30, 0), (-24, \pm 144), (-16, 0), (0, 0)$
15	$(-136, \pm 12376), (-81, \pm 3024), (-45, 0), (-32, 0), (-24, \pm 168), (-17, 0), (0, 0)$
16	$(-153, \pm 16065), (-48, 0), (-34, 0), (-27, \pm 189), (-18, 0), (0, 0)$
17	$(-171, \pm 20520), (-100, \pm 5040), (-51, 0), (-36, 0), (-27, \pm 216), (-19, 0), (0, 0)$
18	$(-190, \pm 25840), (-54, 0), (-38, 0), (-36, \pm 144), (-30, \pm 240), (-20, 0), (0, 0), (10, \pm 960)$
19	$(-210, \pm 32130), (-121, \pm 7920), (-57, 0), (-40, 0), (-30, \pm 270), (-21, 0), (0, 0)$
20	$(-231, \pm 39501), (-60, 0), (-42, 0), (-33, \pm 297), (-22, 0), (0, 0)$

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Elliptic curve

There is an interesting sequence of points appearing in the above table

$$[k, P_k] \in \{[4,(-9,\pm 9)],[5,(-9,\pm 18)],[6,(-12,\pm 24)],[7,(-12,\pm 36)],\ldots\}.$$

A related sequence (A168237) can be found in The On-Line Encyclopedia of Integer Sequences. There is a closed formula given from which we can provide points on the genus 1 model. If k is even, then a point is

$$\left(-\frac{3(k+2)}{2}, \frac{3(k+2)(-k+2)}{4}\right).$$

If k is odd, then we have a point

$$\left(-\frac{3(k+1)}{2}, \frac{3(k-1)(k+1)}{4}\right).$$

Elliptic curve

The torsion subgroup of the Mordell-Weil group of the curve (2) is generated by the points $T_1 = (-k-2, 0)$, $T_2 = (-2k-2, 0)$ and we have that $2T_1 = 2T_2 = \mathcal{O}$, $T_1 + T_2 = (-3k, 0)$. Let us first put the quartic curve into the cubic form

$$y^2 = x^3 + (11k^2 + 18k + 4)x^2 + (36k^4 + 132k^3 + 144k^2 + 48k)x + (36k^6 + 216k^5 + 468k^4 + 432k^3 + 144k^2)$$

by sending the point $(0, 0)$ to infinity using the transformation φ

$$\begin{aligned} x &= \frac{6k^3 + 18k^2 + 12k}{X}, \\ y &= \frac{(6k^3 + 18k^2 + 12k)Y}{X^2}. \end{aligned}$$

Integral points

E.g. when $k = 5$ the integral points are as follows

$$\begin{aligned}
&(0,0),(-7,0),(-12,0),(-15,0),\\
&\pm P_5 = (-9,\pm 18), \pm P_5 \pm (-12,0) = (-21,\pm 126),\\
&\pm 2P_5 \pm (-15,0) = (3,\pm 90),\\
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In case of $k = 20$ we have

$$
\begin{aligned}
&(0,0), (-22,0), (-42,0), (-60,0),\\
&\pm P_{20} = (-33, 297), \pm P_{20} \pm (-22,0) = (-231, \pm 39501).
\end{aligned}
$$

Integral points

If $k = 2t$, then we obtain

$$\begin{aligned}
\pm P_k \pm (-k-2, 0) &= (-2t^2 - 3t - 1, \pm(4t^4 - 5t^2 + 1)),\\
\pm P_k \pm (-3k, 0) &= (-3t - 3/2, \pm(3t^2 - 3/4)),\\
\pm 2P_k \pm (-2k-2) &= \left(\frac{-2t^2 - t}{t - 1/2}, \pm\frac{-4t^3 + 2t^2 + 2t}{t^2 - t + 1/4}\right).
\end{aligned}$$

If $k = 2t + 1$, then we get

$$\begin{aligned}
\pm P_k \pm (-2k-2, 0) &= (-2t^2 - 5t - 3, \pm(4t^4 + 8t^3 + t^2 - 3t)),\\
\pm 2P_k \pm (-k-1, 0) &= \left(\frac{-4t^2 - 8t - 3}{t + 1}, \pm\frac{4t^3 + 8t^2 + 3t}{t^2 + 2t + 1}\right).
\end{aligned}$$

If the rank is 1 and k is large we may expect 8 integral points.

Height bounds

Weierstrass model ($k = 2t$):

$$y^2 = x^3 + (-89856t^4 + 34560t^2 - 6912)x + (7741440t^6 - 331776t^4 - 1658880t^2 + 221184)$$

$$(-3t-3, 3(t+1)(-t+1)) \longrightarrow (-48t^2 + 144t + 48, -3456t^3 + 1728t^2 + 1728t)$$

If Q is an integral point, then $Q = nP_k + T$. Hence $n^2 = \frac{\hat{h}(Q)}{\hat{h}(P_k)}$.
Silverman's bound:

$$\hat{h}(Q) \leq h(Q) + \frac{h(j)}{6} + \frac{h(\Delta)}{6} + 2.14$$

We do not need to apply elliptic logarithm method to have a bound for $h(Q)$! By Runge's method we obtained that $X \leq c_1 k^2$. Hence

$$n^2 \leq \frac{7\log(t) + 18.24}{\hat{h}(P_k)}.$$

Problem by Zhang and Cai

Zhang and Cai deal with the equations

$$\begin{aligned}(x-1)x(x+1)(y-1)y(y+1) &= (z-1)z(z+1),\\ (x-b)x(x+b)(y-b)y(y+b) &= z^2, \text{ where } b \text{ is a positive even number.}\end{aligned}$$

In case of the first equation they prove that there exist infinitely many non-trivial positive integer solutions. In case of the second equation they obtain similar result.

Problem by Zhang and Cai

They also pose two questions related to the previous equations.

Question 1. Are all the nontrivial positive integer solutions of $(x-1)x(x+1)(y-1)y(y+1) = (z-1)z(z+1)$ with $x \leq y$ given by $(F_{2n-1}, F_{2n+1}, F_{2n}^2), n \geq 1$?

Question 2. Are there infinitely many nontrivial positive integer solutions of $(x-b)x(x+b)(y-b)y(y+b) = z^2$ if $b \geq 3$ odd?

Other approach

$$(x-1)x(x+1)(x+k-1)(x+k)(y+k+1) = (z-1)z(z+1)$$

We obtain via Runge's method that

$$\begin{aligned} (x^2+kx-1)^3 < \quad & F(x) \quad < (x^2+kx)^3, \\ (z-1)^3 < \quad & G(z) \quad < (z+1)^3. \end{aligned}$$

Hence

$$(x^2+kx-1)^3 - (z+1)^3 < 0 < (x^2+kx)^3 - (z-1)^3.$$

It follows that $z = x^2 + kx - 1$ or $z = x^2 + kx$.

Large solutions

If $z = x^2 + kx$, then $(k^2 + 2kx + 2x^2 - 2)(k + x)x \Rightarrow x = 0, x = -k$ or $|k| < 2$.

If $z = x^2 + kx - 1$, then $(k^2 - kx - x^2 + 1)(k + x)x \Rightarrow x = 0, x = -k$ or

$$x = -\frac{1}{2}k \pm \frac{1}{2}\sqrt{5k^2 - 4} \Rightarrow k = F_{2n},$$

Therefore

$$x = -F_{2n+1} \text{ or } x = F_{2n}^2.$$

Small solutions

$$(x^2 + kx - 1)^3 < F(x) < (x^2 + kx)^3$$

The second inequality is true if $|k| > 1$. Roots of the polynomial $F(x) - (x^2 + kx - 1)^3$:

$$\begin{aligned}
-\frac{1}{2}\,k - \frac{1}{2}\,\sqrt{3\,k^2 + 2\,\sqrt{k^2+4}k + 4} &\approx -\frac{1}{2}\,k\Big(\sqrt{5}+1\Big),\\
-\frac{1}{2}\,k - \frac{1}{2}\,\sqrt{3\,k^2 - 2\,\sqrt{k^2+4}k + 4} &\approx -k,\\
-\frac{1}{2}\,k + \frac{1}{2}\,\sqrt{3\,k^2 - 2\,\sqrt{k^2+4}k + 4} &\approx \frac{1}{k^3},\\
-\frac{1}{2}\,k + \frac{1}{2}\,\sqrt{3\,k^2 + 2\,\sqrt{k^2+4}k + 4} &\approx \frac{1}{2}\,k\Big(\sqrt{5}-1\Big).
\end{aligned}$$

Question 2.

$$(x-3)x(x+3)(x+k-3)(x+k)(x+k+3) = z^2$$

Puiseux expansion: $-\left(k^2 - 8\,kx - 8\,x^2 + 72\right)(k + 2\,x)$

Large solutions exist$\Rightarrow$ $(k^4 - 12k^3x - 12k^2x^2 + 144k^2 + 1728kx + 1728x^2 + 5184)k^2 = 0$

$$x = -\frac{3\,k^3 \pm 2\,\sqrt{3\,k^2 - 432}\left(k^2 - 36\right) - 432\,k}{6\,(k^2 - 144)}$$

Thus $k = 6\left((2+\sqrt{3})^n + (2-\sqrt{3})^n\right)$. This formula provides infinitely many solutions in $\mathbb{Q}$.