RATIONAL FUNCTION VARIANT OF A PROBLEM OF ERDŐS AND GRAHAM

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Abstract. In this paper we provide bounds for the size of the solutions of the Diophantine equations

$$\begin{aligned} \frac{x(x+1)(x+2)(x+3)(x+4)(x+5)}{(x+a)(x+b)} &= y^2, \\ \frac{x(x+1)(x+2)(x+3)(x+4)(x+5)}{(x+a)(x+b)(x+c)} &= y^3, \\ \frac{x(x+1)(x+2)(x+3)(x+4)(x+5)}{(x+a)(x+b)(x+c)(x+d)} &= y^2, \end{aligned}$$

where $a, b, c, d \in \mathbb{Z}$ are pairwise distinct integers.

1. Introduction

Let us define

$$f(x,k,d) = x(x+d)\cdots(x+(k-1)d),$$

and consider the Diophantine equation

$$f(x,k,d) = y^l. \tag{1}$$

Erdős [6] and independently Rigge [17] proved that the equation $f(x,k,1) = y^2$ has no integer solution. Erdős and Selfridge [7] extended this result when $d = 1$, $x \geq 1$ and $k \geq 2$ and they stated that $f(x,k,1)$ is never a perfect power. This type of Diophantine equations have been studied intensively.

In the first case assume that $l = 2$. Euler solved the equation (1) with $k = 4$ (see [4] pp. 440 and 635) and after that Obláth [16] extended this result to the product of five terms in arithmetic progression, i.e. $k = 5$. If d is a power of a prime number and $k \geq 4$ Saradha and Shorey [20] proved that (1) has no solutions. Laishram and Shorey [14] examined the case where either $d \leq 10^{10}$, or d has at most six prime divisors. Bennett, Bruin, Győry and Hajdu [2] solved (1) when $6 \leq k \leq 11$. Hirata-Kohno, Laishram, Shorey and Tijdeman [13] completely solved the equation (1) with $3 \leq k < 110$. Combining their result with those

2000 *Mathematics Subject Classification.* Primary 11D61; Secondary 11Y50.
Key words and phrases. Diophantine equations.

of Tengely [23] all solutions of (1) with $3 \leq k \leq 100$, $P(b) < k$ are determined, where $P(u)$ denotes the greatest prime factor of u, with the convention $P(1) = 1$.

Now assume for this paragraph that $l \geq 3$. The literature of this equation

$$f(x, k, d) = by^l, \tag{2}$$

with $b > 0$ and $P(b) \leq k$ is very rich. Saradha [19] proved that (2) has no solution with $k \geq 4$. Győry [9] studied the product of two and three consecutive terms in arithmetic progression. Győry, Hajdu and Saradha [11] proved that if $k = 4, 5$ and $\gcd(x, d) = 1$ equation (2) cannot be a perfect power. Hajdu, Tengely and Tijdeman [12] proved that the product of k coprime integers in arithmetic progression cannot be a cube when $2 < k < 39$. If $3 < k < 35$ and $\gcd(x, d) = 1$ Győry, Hajdu and Pintér [10] proved that for any positive integers x, d and k the product $f(x, k, d)$ cannot be a perfect power.

Erdős and Graham [5] asked if the Diophantine equation

$$\prod_{i=1}^{r} f(x_i, k_i, 1) = y^2$$

has, for fixed $r \geq 1$ and $\{k_1, k_2, \ldots, k_r\}$ with $k_i \geq 4$ for $i = 1, 2, \ldots, r$, at most finitely many solutions in positive integers $(x_1, x_2, \ldots, x_r, y)$ with $x_i + k_i \leq x_{i+1}$ for $1 \leq i \leq r - 1$. Skałba [21] provided a bound for the smallest solution and estimated the number of solutions below a given bound. Ulas [25] gave a counterexample when either $r = k_i = 4$, or $r \geq 6$ and $k_i = 4$. Bauer and Bennett [1] extended this result to the cases $r = 3$ and $r = 5$. In the case $k_i = 5$ and $r \geq 5$ Bennett and Van Luijk [3] constructed an infinite family such that the product $\prod_{i=1}^{r} f(x_i, k_i, 1)$ is always a perfect square. Luca and Walsh [15] considered the case $(r, k_i) = (2, 4)$.

In our previous paper [24] we considered the equation

$$\frac{x(x+1)(x+2)(x+3)}{(x+a)(x+b)} = y^2$$

where $a, b \in \mathbb{Z}, a \neq b$ are parameters. We provided bounds for the size of solutions and an algorithm to determine all solutions $(x, y) \in \mathbb{Z}^2$. The proof based on Runge's method and the result of Sankaranarayanan and Saradha [18].

In this paper we extended this latter result and study the following three Diophantine equations

$$\begin{aligned}\frac{x(x+1)(x+2)(x+3)(x+4)(x+5)}{(x+a)(x+b)} &= y^2,\\ \frac{x(x+1)(x+2)(x+3)(x+4)(x+5)}{(x+a)(x+b)(x+c)} &= y^3,\\ \frac{x(x+1)(x+2)(x+3)(x+4)(x+5)}{(x+a)(x+b)(x+c)(x+d)} &= y^2,\end{aligned}$$

where $a, b, c, d \in \mathbb{Z}$ are pairwise distinct integers such that $a, b, c, d \notin \{0, 1, 2, 3, 4, 5\}$. Bounds for the solutions of these equations are provided in the following three theorems.

Theorem 1. *If $(x, y) \in \mathbb{Z}^2$ is a solution of the Diophantine equation*

$$\frac{x(x+1)(x+2)(x+3)(x+4)(x+5)}{(x+a)(x+b)} = y^2,$$

then either

$$x \mid \left(3\,a^2 + 2\,ab + 3\,b^2 - 30\,a - 30\,b + 115\right)^2 ab$$

or

$$|x| \leq 16t^3 + 440t^2,$$

where $t = \max\{|a|, |b|\}$.

Theorem 2. *If $(x, y) \in \mathbb{Z}^2$ is a solution of the Diophantine equation*

$$\frac{x(x+1)(x+2)(x+3)(x+4)(x+5)}{(x+a)(x+b)(x+c)} = y^3,$$

then either

$$x \mid (a+b+c-15)^3 abc$$

or

$$|x| \leq 6t^2 + 68t,$$

where $t = \max\{|a|, |b|, |c|\}$.

Theorem 3. *If $(x, y) \in \mathbb{Z}^2$ is a solution of the Diophantine equation*

$$\frac{x(x+1)(x+2)(x+3)(x+4)(x+5)}{(x+a)(x+b)(x+c)(x+d)} = y^2,$$

then either

$$x \mid (a+b+c+d-15)^2 abcd$$

or

$$|x| \leq 12t^2 + 132t,$$

where $t = \max\{|a|, |b|, |c|, |d|\}$.

We will use the following result of Fujiwara [8] to prove our statements.

Lemma 1. *Given* $p(z)=\sum_{i=0}^{n} a_i z^i, a_n\neq 0$. *Then*

$$\max\{|\zeta| : p(\zeta)=0\} \leq 2\max\left\{\left|\frac{a_{n-1}}{a_n}\right|, \left|\frac{a_{n-2}}{a_n}\right|^{1/2}, \dots, \left|\frac{a_0}{a_n}\right|^{1/n}\right\}.$$

2. Proof of Theorem 1

Now we deal with the equation

$$F(x)=\frac{x(x+1)(x+2)(x+3)(x+4)(x+5)}{(x+a)(x+b)}=y^2. \tag{3}$$

The polynomial part of the Puiseux expansion of $F(x)^{1/2}$ is

$$P(x)=x^2-\frac{a+b-15}{2}x+\frac{3a^2+2ab+3b^2-30a-30b+115}{8}.$$

Let

$$A(x)=x(x+1)(x+2)(x+3)(x+4)(x+5)-(x+a)(x+b)\left(P(x)-\frac{1}{8}\right)^2$$

and

$$B(x)=x(x+1)(x+2)(x+3)(x+4)(x+5)-(x+a)(x+b)\left(P(x)+\frac{1}{8}\right)^2$$

We have that $\deg A=\deg B=4$ and the leading coefficient of A is $1/4$ and the leading coefficient of B is $-1/4$. Denote by I_A an interval containing all zeroes of the polynomial $A(x)$ and by I_B the interval containing all zeroes of $B(x)$. We observe that if $x<\min\{a,b\}$ or $x>\max\{a,b\}$ and we also have that $x\notin I_A, x\notin I_B$, then

$$\frac{A(x)}{(x+a)(x+b)} \quad \text{and} \quad \frac{B(x)}{(x+a)(x+b)}$$

have opposite signs. Therefore there are two possibilities. Either

$$\begin{aligned} F(x)-\left(P(x)-\frac{1}{8}\right)^2 &< 0,\\ F(x)-\left(P(x)+\frac{1}{8}\right)^2 &> 0 \end{aligned}$$

or

$$\begin{aligned} F(x) - \left(P(x) - \frac{1}{8}\right)^2 &> 0, \\ F(x) - \left(P(x) + \frac{1}{8}\right)^2 &< 0. \end{aligned}$$

We only handle the first case, the second case is very similar. Here we obtain that

$$\left(P(x) + \frac{1}{8}\right)^2 < F(x) = y^2 < \left(P(x) - \frac{1}{8}\right)^2.$$

Hence

$$(8P(x) + 1)^2 < (8y)^2 < (8P(x) - 1)^2.$$

The polynomial $8P(x)$ has integral coefficients, so if x is an integer, then $8P(x)$ is an integer as well. For a fixed integer x there is only one square integer between $(8P(x) + 1)^2$ and $(8P(x) - 1)^2$, it is $64P(x)^2$. Thus $y = P(x)$ and x divides the constant term of the polynomial $64x(x+1)(x+2)(x+3)(x+4)(x+5) - 64(x+a)(x+b)P(x)^2$, that is x divides

$$\left(3\,a^2 + 2\,ab + 3\,b^2 - 30\,a - 30\,b + 115\right)^2 ab.$$

It remains to provide an upper bound for the size of roots of $A(x) = \frac{1}{4}x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$ and $B(x) = -\frac{1}{4}x^4 + b_3x^3 + b_2x^2 + b_1x + b_0$. Let $t = \max\{|a|, |b|\}$. We have that

$$\begin{aligned} |4a_3| &\leq 8t^3 + 60t^2 + 114t + 45, \\ |4a_2| &\leq \frac{15}{4}\,t^4 + 60\,t^3 + 450\,t^2 + 855\,t + \frac{1135}{4}, \\ |4a_1| &\leq \frac{9}{4}\,t^5 + 45\,t^4 + 282\,t^3 + 855\,t^2 + \frac{3249}{2}\,t + 480, \\ |4a_0| &\leq 4\,t^6 + 60\,t^5 + 339\,t^4 + 855\,t^3 + \frac{3249}{4}\,t^2. \end{aligned}$$

Similarly we obtain that

$$\begin{aligned} |4b_3| &\leq 8t^3 + 60t^2 + 116t + 30, \\ |4b_2| &\leq \frac{15}{4}\,t^4 + 60\,t^3 + 450\,t^2 + 870\,t + 255, \\ |4b_1| &\leq \frac{9}{4}\,t^5 + 45\,t^4 + 283\,t^3 + 870\,t^2 + 1682\,t + 480, \\ |4b_0| &\leq 4\,t^6 + 60\,t^5 + 341\,t^4 + 870\,t^3 + 841\,t^2. \end{aligned}$$

By Fujiwara's result it follows that

$$\max\{|\zeta| : A(\zeta) = 0 \text{ or } B(\zeta) = 0\} \leq 16t^3 + 440t^2.$$

3. Proof of Theorem 2

Now, we consider the equation

$$\frac{x(x+1)(x+2)(x+3)(x+4)(x+5)}{(x+a)(x+b)(x+c)} = y^3, \tag{4}$$

where $a, b, c \in \mathbb{Z}$ are pairwise distinct integers with $a, b, c \notin \{0, 1, 2, 3, 4, 5\}$. The polynomial part of the Puiseux expansion of

$$\left(\frac{x(x+1)(x+2)(x+3)(x+4)(x+5)}{(x+a)(x+b)(x+c)}\right)^{1/3}$$

is $P(x) = x + 5 - \frac{a+b+c}{3}$. Define

$$A(x) = x(x+1)(x+2)(x+3)(x+4)(x+5) - (x+a)(x+b)(x+c)\left(P(x) - \frac{1}{3}\right)^3$$

and

$$B(x) = x(x+1)(x+2)(x+3)(x+4)(x+5) - (x+a)(x+b)(x+c)\left(P(x) + \frac{1}{3}\right)^3.$$

We obtain that $\deg A = \deg B = 5$ and the leading coefficient of A is 1 and the leading coefficient of B is -1. Therefore

$$\frac{A(x)}{(x+a)(x+b)(x+c)} \text{ and } \frac{B(x)}{(x+a)(x+b)(x+c)}$$

have opposite signs if $|x|$ is larger than the maximum of the zeroes of $A(x)B(x)$ in absolute value. The following two possibilities can occur. Either

$$\left(P(x) - \frac{1}{3}\right)^3 < \frac{x(x+1)(x+2)(x+3)(x+4)(x+5)}{(x+a)(x+b)(x+c)} < \left(P(x) + \frac{1}{3}\right)^3$$

or

$$\left(P(x) + \frac{1}{3}\right)^3 < \frac{x(x+1)(x+2)(x+3)(x+4)(x+5)}{(x+a)(x+b)(x+c)} < \left(P(x) - \frac{1}{3}\right)^3.$$

In a similar way than in the proof of Theorem 1 one gets that $y = P(x) = x + 5 - \frac{a+b+c}{3}$. Hence x divides the constant coefficient of the polynomial

$$27x(x+1)(x+2)(x+3)(x+4)(x+5) - 27(x+a)(x+b)(x+c)P(x)^3,$$

that is

$$x \mid (a+b+c-15)^3 abc.$$

It remains to determine a bound for the maximum of the zeroes of $A(x)B(x)$ in absolute value. We apply Fujiwara's result to obtain such a bound. We have that $A(x) = x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$ and

$B(x) = -x^5 + b_4x^4 + b_3x^3 + b_2x^2 + b_1x + b_0$. Let $t = \max\{|a|, |b|, |c|\}$. First we compute bounds for the absolute value of the coefficients of $A(x)$ and $B(x)$. These are as follows

$$\begin{aligned}
|a_4| &\leq 3t^2 + 14t + 59/3,\\
|a_3| &\leq 16/9t^3 + 28t^2 + 392/3t + 3331/27,\\
|a_2| &\leq 29/9t^4 + 112/3t^3 + 392/3t^2 + 2744/9t + 274,\\
|a_1| &\leq 16/9t^5 + 70/3t^4 + 392/3t^3 + 2744/9t^2 + 120,\\
|a_0| &\leq t^6 + 14t^5 + 196/3t^4 + 2744/27t^3
\end{aligned}$$

and

$$\begin{aligned}
|b_4| &\leq 3t^2 + 16t + 1/3,\\
|b_3| &\leq 16/9t^3 + 32t^2 + 512/3t + 1979/27,\\
|b_2| &\leq 29/9t^4 + 128/3t^3 + 512/3t^2 + 4096/9t + 274,\\
|b_1| &\leq 16/9t^5 + 80/3t^4 + 512/3t^3 + 4096/9t^2 + 120,\\
|b_0| &\leq t^6 + 16t^5 + 256/3t^4 + 4096/27t^3.
\end{aligned}$$

One needs to establish a bound for $|a_{5-i}|^{1/i}$ and $|b_{5-i}|^{1/i}, i = 1, 2, \ldots, 5$. One has that $\max\{|a_{5-i}|^{1/i}, |b_{5-i}|^{1/i}\} \leq 3t^2 + 34t$. Thus Fujiwara's bound implies that $|x| \leq 6t^2 + 68t$.

4. Proof of Theorem 3

Let us study the Diophantine equation

$$\frac{x(x+1)(x+2)(x+3)(x+4)(x+5)}{(x+a)(x+b)(x+c)(x+d)} = y^2, \tag{5}$$

where $a, b, c, d \in \mathbb{Z}$ are pairwise distinct integers with $a, b, c, d \notin \{0, 1, 2, 3, 4, 5\}$. The polynomial part of the Puiseux expansion of

$$\left(\frac{x(x+1)(x+2)(x+3)(x+4)(x+5)}{(x+a)(x+b)(x+c)(x+d)}\right)^{1/2}$$

is $P(x) = x + \frac{15-(a+b+c+d)}{2}$. Let

$$A(x) = x(x+1)(x+2)(x+3)(x+4)(x+5) - (x+a)(x+b)(x+c)(x+d)\left(P(x) - \frac{1}{2}\right)^2$$

and

$$B(x) = x(x+1)(x+2)(x+3)(x+4)(x+5) - (x+a)(x+b)(x+c)(x+d)\left(P(x) + \frac{1}{2}\right)^2.$$

The degree of $A(x)$ is 5 and the leading coefficient is 1, the degree of $B(x)$ is also 5 and the leading coefficient is -1. So one has that

$$\frac{A(x)}{(x+a)(x+b)(x+c)(x+d)} \text{ and } \frac{B(x)}{(x+a)(x+b)(x+c)(x+d)}$$

have opposite signs if $|x|$ is larger than the maximum of the zeroes of $A(x)B(x)$ in absolute value. It follows that either

$$\left(P(x)-\frac{1}{2}\right)^2 < \frac{x(x+1)(x+2)(x+3)(x+4)(x+5)}{(x+a)(x+b)(x+c)(x+d)} < \left(P(x)+\frac{1}{2}\right)^2$$

or

$$\left(P(x)+\frac{1}{2}\right)^2 < \frac{x(x+1)(x+2)(x+3)(x+4)(x+5)}{(x+a)(x+b)(x+c)(x+d)} < \left(P(x)-\frac{1}{2}\right)^2.$$

We conclude that if $|x|$ is large, then $y = P(x) = x + \frac{15-(a+b+c+d)}{2}$ and x divides the constant term of the polynomial

$$4x(x+1)(x+2)(x+3)(x+4)(x+5)-4(x+a)(x+b)(x+c)(x+d)P(x)^2.$$

That is

$$x \mid (a+b+c+d-15)^2abcd.$$

Now we compute bounds for $|a_i|$ and $|b_i|, i = 0, 1, \dots, 4$, where $A(x) = x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$ and $B(x) = -x^5 + b_4x^4 + b_3x^3 + b_2x^2 + b_1x + b_0$. Let $t = \max\{|a|, |b|, |c|, |d|\}$. We have that

$$\begin{aligned}
|a_4| &\leq 6t^2 + 28t + 36,\\
|a_3| &\leq 6t^3 + 28t^2 + 196t + 225,\\
|a_2| &\leq 9t^4 + 112t^3 + 294t^2 + 274,\\
|a_1| &\leq 12t^5 + 98t^4 + 196t^3 + 120,\\
|a_0| &\leq 4t^6 + 28t^5 + 49t^4
\end{aligned}$$

and

$$\begin{aligned}
|b_4| &\leq 6t^2 + 32t + 21,\\
|b_3| &\leq 6t^3 + 32t^2 + 256t + 225,\\
|b_2| &\leq 9t^4 + 128t^3 + 384t^2 + 274,\\
|b_1| &\leq 12t^5 + 112t^4 + 256t^3 + 120,\\
|b_0| &\leq 4t^6 + 32t^5 + 64t^4.
\end{aligned}$$

One obtains that $\max\{|a_{5-i}|^{1/i}, |b_{5-i}|^{1/i}\} \leq 6t^2 + 66t$. Thus Fujiwara's bound implies that $|x| \leq 12t^2 + 132t$.

5. Numerical results

In this section we provide complete solutions of the considered three Diophantine equations for certain values of the parameters. We wrote Sage [22] codes to compute all solutions $(x, y) \in \mathbb{Z}^2$ of the concrete equations. It can be downloaded from
`http://www.math.unideb.hu/~tengely/RatFunErdosGraham.sage`.

Theorem 4. *Let $a < b$ integers such that $a, b \in \{-10, -9, \ldots, 14, 15\} \setminus \{0, 1, 2, 3, 4, 5\}$. The pairs $[a, b]$ for which equation (3) has a non-trivial solution are given by*

$[a, b]$	*list of solutions* $[x, y]$
$[-10, -8]$	$[[0,0], [3,24], [-1,0], [-5,0], [-4,0], [-3,0], [-2,0]]$
$[-10, -6]$	$[[0,0], [1,4], [-2,0], [-5,0], [-4,0], [-3,0], [-1,0]]$
$[-9, -7]$	$[[0,0], [2,12], [-2,0], [-5,0], [-4,0], [-3,0], [-1,0]]$
$[-9, -6]$	$[[0,0], [-2,0], [-6,2], [-5,0], [-4,0], [-3,0], [-1,0]]$
$[-7, -3]$	$[[0,0], [-1,0], [-7,6], [-5,0], [-4,0], [-3,0], [-2,0]]$
$[-6, -5]$	$[[0,0], [1,6], [-1,0], [-5,0], [-4,0], [-3,0], [-2,0]]$
$[-6, -2]$	$[[0,0], [1,12], [-2,0], [-8,12], [-5,0], [-4,0], [-3,0], [-1,0]]$
$[-4, -2]$	$[[0,0], [-1,0], [-10,30], [-6,3], [-5,0], [-4,0], [-3,0], [-2,0]]$
$[-4, 7]$	$[[0,0], [-1,0], [-10,60], [-5,0], [-4,0], [-3,0], [-2,0]]$
$[-2, 9]$	$[[0,0], [5,60], [-2,0], [-5,0], [-4,0], [-3,0], [-1,0]]$
$[7, 9]$	$[[0,0], [1,3], [5,30], [-1,0], [-5,0], [-4,0], [-3,0], [-2,0]]$
$[7, 11]$	$[[0,0], [3,12], [-2,0], [-6,12], [-5,0], [-4,0], [-3,0], [-1,0]]$
$[8, 12]$	$[[0,0], [2,6], [-1,0], [-5,0], [-4,0], [-3,0], [-2,0]]$
$[10, 11]$	$[[0,0], [-1,0], [-6,6], [-5,0], [-4,0], [-3,0], [-2,0]]$
$[11, 14]$	$[[0,0], [1,2], [-2,0], [-5,0], [-4,0], [-3,0], [-1,0]]$
$[11, 15]$	$[[0,0], [-2,0], [-6,4], [-5,0], [-4,0], [-3,0], [-1,0]]$
$[12, 14]$	$[[0,0], [-2,0], [-7,12], [-5,0], [-4,0], [-3,0], [-1,0]]$
$[13, 15]$	$[[0,0], [-1,0], [-8,24], [-5,0], [-4,0], [-3,0], [-2,0]]$

Theorem 5. *Let $a < b < c$ integers such that $a, b, c \in \{-7, -6, \ldots, 12\} \setminus \{0, 1, 2, 3, 4, 5\}$. The triples $[a, b, c]$ for which equation (4) has a non-trivial solution are given by*

$[a, b, c]$	*list of solutions* $[x, y]$
$[-7, -6, -4]$	$[[0,0], [1,-2], [-1,0], [-8,-2], [-5,0], [-4,0], [-3,0], [-2,0]]$
$[-7, -5, -1]$	$[[0,0], [-2,0], [-9,-3], [-5,0], [-4,0], [-3,0], [-1,0]]$
$[-7, -2, 12]$	$[[0,0], [-2,0], [-7,2], [-5,0], [-4,0], [-3,0], [-1,0]]$
$[-7, 7, 12]$	$[[0,0], [2,-2], [-2,0], [-5,0], [-4,0], [-3,0], [-1,0]]$
$[-7, 9, 11]$	$[[0,0], [1,-1], [-2,0], [-5,0], [-4,0], [-3,0], [-1,0]]$
$[-6, -4, 12]$	$[[0,0], [-2,0], [-6,1], [-5,0], [-4,0], [-3,0], [-1,0]]$
$[-6, -3, 8]$	$[[0,0], [1,2], [-1,0], [-5,0], [-4,0], [-3,0], [-2,0]]$
$[-6, 6, 10]$	$[[0,0], [4,-6], [-1,0], [-5,0], [-4,0], [-3,0], [-2,0]]$
$[-5, -1, 7]$	$[[0,0], [-2,0], [-9,-6], [-5,0], [-4,0], [-3,0], [-1,0]]$
$[-5, -1, 11]$	$[[0,0], [-1,0], [-9,6], [-5,0], [-4,0], [-3,0], [-2,0]]$
$[-4, -3, -2]$	$[[0,0], [-1,0], [-6,-1], [-5,0], [-4,0], [-3,0], [-2,0]]$
$[-4, -3, 7]$	$[[0,0], [-1,0], [-6,2], [-5,0], [-4,0], [-3,0], [-2,0]]$
$[-3, 8, 11]$	$[[0,0], [-1,0], [-6,-2], [-5,0], [-4,0], [-3,0], [-2,0]]$
$[-2, 6, 10]$	$[[0,0], [4,6], [-2,0], [-5,0], [-4,0], [-3,0], [-1,0]]$
$[-2, 8, 9]$	$[[0,0], [1,-2], [-1,0], [-5,0], [-4,0], [-3,0], [-2,0]]$
$[6, 10, 12]$	$[[0,0], [4,3], [-2,0], [-5,0], [-4,0], [-3,0], [-1,0]]$
$[7, 8, 9]$	$[[0,0], [1,1], [-2,0], [-5,0], [-4,0], [-3,0], [-1,0]]$
$[9, 11, 12]$	$[[0,0], [3,2], [-1,0], [-6,2], [-5,0], [-4,0], [-3,0], [-2,0]]$

Theorem 6. *Let $a < b < c < d$ integers such that $a, b, c, d \in \{-7, -6, \ldots, 12\} \setminus \{0, 1, 2, 3, 4, 5\}$. The tuples $[a, b, c, d]$ for which equation (5) has a non-trivial solution are given by*

$[a, b, c, d]$	*list of solutions* $[x, y]$
$[-7, -6, -5, 7]$	$[[0, 0], [-1, 0], [-9, 3], [-5, 0], [-4, 0], [-3, 0], [-2, 0]]$
$[-7, -6, -4, -3]$	$[[0, 0], [1, 2], [-1, 0], [-5, 0], [-4, 0], [-3, 0], [-2, 0]]$
$[-7, -6, -4, 6]$	$[[0, 0], [-1, 0], [-8, 2], [-5, 0], [-4, 0], [-3, 0], [-2, 0]]$
$[-7, -6, 10, 11]$	$[[0, 0], [-2, 0], [-8, 4], [-5, 0], [-4, 0], [-3, 0], [-1, 0]]$
$[-7, -5, -1, 6]$	$[[0, 0], [-2, 0], [-9, 3], [-5, 0], [-4, 0], [-3, 0], [-1, 0]]$
$[-7, -5, 6, 10]$	$[[0, 0], [4, 12], [-2, 0], [-5, 0], [-4, 0], [-3, 0], [-1, 0]]$
$[-7, -4, -1, 12]$	$[[0, 0], [2, 6], [-2, 0], [-5, 0], [-4, 0], [-3, 0], [-1, 0]]$
$[-7, -4, 7, 11]$	$[[0, 0], [3, 6], [-2, 0], [-5, 0], [-4, 0], [-3, 0], [-1, 0]]$
$[-7, -4, 7, 12]$	$[[0, 0], [2, 2], [-2, 0], [-5, 0], [-4, 0], [-3, 0], [-1, 0]]$
$[-7, -3, -2, 6]$	$[[0, 0], [-2, 0], [-7, 2], [-5, 0], [-4, 0], [-3, 0], [-1, 0]]$
$[-7, -3, 6, 12]$	$[[0, 0], [2, 3], [-2, 0], [-5, 0], [-4, 0], [-3, 0], [-1, 0]]$
$[-7, -3, 8, 11]$	$[[0, 0], [-2, 0], [-7, 3], [-5, 0], [-4, 0], [-3, 0], [-1, 0]]$
$[-7, -2, 9, 11]$	$[[0, 0], [1, 1], [-2, 0], [-5, 0], [-4, 0], [-3, 0], [-1, 0]]$
$[-7, -2, 9, 12]$	$[[0, 0], [-2, 0], [-7, 2], [-5, 0], [-4, 0], [-3, 0], [-1, 0]]$
$[-7, -1, 8, 12]$	$[[0, 0], [-2, 0], [-7, 3], [-5, 0], [-4, 0], [-3, 0], [-1, 0]]$
$[-7, 6, 9, 12]$	$[[0, 0], [-2, 0], [-7, 6], [-5, 0], [-4, 0], [-3, 0], [-1, 0]]$
$[-6, -5, 7, 8]$	$[[0, 0], [-2, 0], [-9, 12], [-5, 0], [-4, 0], [-3, 0], [-1, 0]]$
$[-6, -5, 10, 11]$	$[[0, 0], [4, 12], [-2, 0], [-9, 12], [-5, 0], [-4, 0], [-3, 0], [-1, 0]]$
$[-6, -5, 11, 12]$	$[[0, 0], [3, 4], [-2, 0], [-5, 0], [-4, 0], [-3, 0], [-1, 0]]$
$[-6, -4, 7, 9]$	$[[0, 0], [-1, 0], [-10, 15], [-5, 0], [-4, 0], [-3, 0], [-2, 0]]$
$[-6, -4, 7, 12]$	$[[0, 0], [-2, 0], [-6, 1], [-5, 0], [-4, 0], [-3, 0], [-1, 0]]$
$[-6, -4, 8, 9]$	$[[0, 0], [-2, 0], [-6, 1], [-5, 0], [-4, 0], [-3, 0], [-1, 0]]$
$[-6, -3, 7, 8]$	$[[0, 0], [1, 1], [-1, 0], [-5, 0], [-4, 0], [-3, 0], [-2, 0]]$
$[-6, -3, 8, 12]$	$[[0, 0], [2, 3], [-2, 0], [-5, 0], [-4, 0], [-3, 0], [-1, 0]]$
$[-6, -2, -1, 7]$	$[[0, 0], [-1, 0], [-8, 4], [-5, 0], [-4, 0], [-3, 0], [-2, 0]]$
$[-6, -2, 9, 12]$	$[[0, 0], [-2, 0], [-8, 6], [-5, 0], [-4, 0], [-3, 0], [-1, 0]]$
$[-5, -3, -1, 8]$	$[[0, 0], [-1, 0], [-9, 6], [-5, 0], [-4, 0], [-3, 0], [-2, 0]]$
$[-5, -3, 8, 9]$	$[[0, 0], [1, 1], [-1, 0], [-5, 0], [-4, 0], [-3, 0], [-2, 0]]$
$[-5, -1, 6, 8]$	$[[0, 0], [-1, 0], [-9, 12], [-5, 0], [-4, 0], [-3, 0], [-2, 0]]$
$[-5, -1, 10, 12]$	$[[0, 0], [-2, 0], [-9, 12], [-5, 0], [-4, 0], [-3, 0], [-1, 0]]$
$[-4, -3, 7, 8]$	$[[0, 0], [-1, 0], [-6, 2], [-5, 0], [-4, 0], [-3, 0], [-2, 0]]$
$[-4, -3, 8, 10]$	$[[0, 0], [-1, 0], [-6, 1], [-5, 0], [-4, 0], [-3, 0], [-2, 0]]$
$[-4, -3, 9, 11]$	$[[0, 0], [1, 1], [-2, 0], [-5, 0], [-4, 0], [-3, 0], [-1, 0]]$
$[-4, -2, -1, 9]$	$[[0, 0], [5, 30], [-1, 0], [-5, 0], [-4, 0], [-3, 0], [-2, 0]]$
$[-4, -2, 6, 9]$	$[[0, 0], [-1, 0], [-10, 15], [-5, 0], [-4, 0], [-3, 0], [-2, 0]]$
$[-4, -2, 9, 11]$	$[[0, 0], [5, 15], [-2, 0], [-5, 0], [-4, 0], [-3, 0], [-1, 0]]$
$[-4, -1, 7, 9]$	$[[0, 0], [5, 15], [-1, 0], [-5, 0], [-4, 0], [-3, 0], [-2, 0]]$
$[-4, 6, 7, 9]$	$[[0, 0], [-1, 0], [-10, 30], [-5, 0], [-4, 0], [-3, 0], [-2, 0]]$
$[-3, -2, 8, 9]$	$[[0, 0], [1, 2], [-1, 0], [-5, 0], [-4, 0], [-3, 0], [-2, 0]]$
$[-3, -2, 8, 11]$	$[[0, 0], [-1, 0], [-6, 1], [-5, 0], [-4, 0], [-3, 0], [-2, 0]]$
$[-3, -2, 10, 11]$	$[[0, 0], [4, 12], [-2, 0], [-5, 0], [-4, 0], [-3, 0], [-1, 0]]$
$[-3, -1, 6, 10]$	$[[0, 0], [4, 12], [-1, 0], [-5, 0], [-4, 0], [-3, 0], [-2, 0]]$
$[-3, 6, 8, 10]$	$[[0, 0], [4, 6], [-2, 0], [-5, 0], [-4, 0], [-3, 0], [-1, 0]]$
$[-2, 6, 7, 11]$	$[[0, 0], [3, 4], [-2, 0], [-5, 0], [-4, 0], [-3, 0], [-1, 0]]$
$[-2, 10, 11, 12]$	$[[0, 0], [4, 3], [-1, 0], [-5, 0], [-4, 0], [-3, 0], [-2, 0]]$
$[-1, 6, 10, 12]$	$[[0, 0], [4, 3], [-2, 0], [-5, 0], [-4, 0], [-3, 0], [-1, 0]]$
$[-1, 7, 8, 12]$	$[[0, 0], [2, 2], [-2, 0], [-5, 0], [-4, 0], [-3, 0], [-1, 0]]$
$[-1, 9, 11, 12]$	$[[0, 0], [3, 2], [-1, 0], [-5, 0], [-4, 0], [-3, 0], [-2, 0]]$
$[8, 9, 11, 12]$	$[[0, 0], [-2, 0], [-6, 2], [-5, 0], [-4, 0], [-3, 0], [-1, 0]]$

Acknowledgement. Szabolcs Tengely was supported by the European Union and the State of Hungary, co-financed by the European Social Fund in the framework of TÁMOP 4.2.4. A/2-11-1-2012-0001 "National Excellence Program".

Nóra Varga was supported by the European Union and the State of Hungary, co-financed by the European Social Fund in the framework of TÁMOP 4.2.4. A/2-11-1-2012-0001 "National Excellence Program".

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