Lucas sequences and infinite sums

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Summary of the talk

The Fibonacci number 89

Let F_n be the Fibonacci sequence, that is

$$\begin{aligned} F_0 &= 0, \\ F_1 &= 1, \\ F_n &= F_{n-1} + F_{n-2}, \quad n \geq 2. \end{aligned}$$

The first few values are $0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89$.

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In 1953 Stancliff noted an interesting property of the Fibonacci number $F_{11} = 89$. One has that

$$\frac{1}{89} = \frac{0}{10} + \frac{1}{10^2} + \frac{1}{10^3} + \frac{2}{10^4} + \frac{3}{10^5} + \frac{5}{10^6} + \dots,$$

where in the numerators the elements of the Fibonacci sequence appear.

Decimal expansion

In 1980 Winans investigated the related sums

$$\sum_{k=0}^{\infty} \frac{F_{\alpha k}}{10^{k+1}}$$

for certain values of α. In 1981 Hudson and Winans provided a complete characterization of all decimal fractions that can be approximated by sums of the type

$$\frac{1}{F_\alpha} \sum_{k=1}^{n} \frac{F_{\alpha k}}{10^{l(k+1)}}, \quad \alpha, l \geq 1.$$

Long proved a general identity for binary recurrence sequences from which one obtains e.g.

$$\frac{1}{9899} = \sum_{k=0}^{\infty} \frac{F_k}{10^{2(k+1)}}, \quad \frac{1}{109} = \sum_{k=0}^{\infty} \frac{F_k}{(-10)^{k+1}}.$$

Different bases

In case of different bases characterizations were obtained by Jia Sheng Lee and by Köhler and by Jin Zai Lee and Jia Sheng Lee. Here we state a result by Köhler that we will use later

Theorem A

Let A, B, a_0, a_1 be arbitrary complex numbers. Define the sequence $\{a_n\}$ by the recursion $a_{n+1} = Aa_n + Ba_{n-1}$. Then the formula

$$\sum_{k=0}^{\infty} \frac{a_k}{x^{k+1}} = \frac{a_0 x - Aa_0 + a_1}{x^2 - Ax - B}$$

holds for all complex x such that $|x|$ is larger than the absolute values of the zeros of $x^2 - Ax - B$.

Lucas sequences

Let P and Q be non-zero relatively prime integers. The Lucas sequence $\{U_n(P,Q)\}$ is defined by

$$U_0 = 0, U_1 = 1 \text{ and } U_n = PU_{n-1} - QU_{n-2}, \text{ if } n \geq 2.$$

In this talk we deal with the determination of all integers $x \geq 2$ for which there exists an $n \geq 0$ such that

$$\frac{1}{U_n} = \sum_{k=1}^{\infty} \frac{U_{k-1}}{x^k}, \tag{1}$$

where U_n is a Lucas sequence with some given P and Q.

Fibonacci case

In case of $P = 1, Q = -1$ one gets the Fibonacci sequence. De Weger computed all $x \geq 2$ in case of the Fibonacci sequence, the solutions are as follows

$$\frac{1}{F_1} = \frac{1}{F_2} = \frac{1}{1} = \sum_{k=1}^{\infty} \frac{F_{k-1}}{2^k}, \qquad \frac{1}{F_5} = \frac{1}{5} = \sum_{k=1}^{\infty} \frac{F_{k-1}}{3^k},$$

$$\frac{1}{F_{10}} = \frac{1}{55} = \sum_{k=1}^{\infty} \frac{F_{k-1}}{8^k}, \qquad \frac{1}{F_{11}} = \frac{1}{89} = \sum_{k=1}^{\infty} \frac{F_{k-1}}{10^k}.$$

Statements

Theorem

Let $\{U_n(P,Q)\}$ be a Lucas sequence with $Q \in \{\pm 1\}$. Then equation (1) possesses only finitely many solutions in n, x which can be effectively determined.

The proof of the above Theorem provides an algorithm to determine all solutions of equation (1). Following this algorithm we obtained numerical results.

Numerical result

Theorem

Let $\{U_n(P,Q)\}$ be a Lucas sequence with $-10 \le P \le 10$, $Q \in \{\pm 1\}$ and $(P,Q) \neq (-2,1),(2,1)$. Then equation (1) has the following solutions

$$
\begin{aligned}
(P,Q,n,x) \in \{&(-3,1,5,6),(-1,-1,5,2),(-1,-1,11,9),(1,-1,1,2),\\
&(1,-1,2,2),(1,-1,5,3),(1,-1,10,8),(1,-1,11,10),(2,-1,2,3),\\
&(3,-1,2,4),(3,1,1,3),(3,1,5,9),(4,-1,2,5),(4,-1,10,647),\\
&(4,1,1,4),(5,-1,2,6),(5,1,1,5),(6,-1,2,7),(6,1,1,6),\\
&(7,-1,2,8),(7,1,1,7),(8,-1,2,9),(8,1,1,8),(9,-1,2,10),\\
&(9,1,1,9),(10,-1,2,11),(10,1,1,10)\}.
\end{aligned}
$$

Associated Lucas sequences

The Lucas sequences $\{U_n(P,Q)\}$ and associated Lucas sequences $\{V_n(P,Q)\}$ are defined by the same linear recurrent relation with $P,Q \in \mathbb{Z} \setminus \{0\}$ but different initial terms:

$$\begin{aligned} U_0 = 0, U_1 = 1 \text{ and } \quad & U_n = PU_{n-1} - QU_{n-2}, \text{ if } n \geq 2, \\ V_0 = 2, V_1 = P \text{ and } \quad & V_n = PV_{n-1} - QV_{n-2}, \text{ if } n \geq 2. \end{aligned}$$

Terms of Lucas sequences and associated Lucas sequences satisfy the following identity

$$V_n^2 - DU_n^2 = 4Q^n, \tag{2}$$

where $D = P^2 - 4Q$.

Ternary quadratic equations

To determine the appropriate Thue equations we use parametric solutions of ternary quadratic equations.

Lemma (Alekseyev-Tengely)

Let A, B, C be non-zero integers and let (x_0, y_0, z_0) with $z_0 \neq 0$ be a particular non-trivial integer solution to the Diophantine equation $Ax^2 + By^2 + Cz^2 = 0$. Then its general integer solution is given by

$$(x, y, z) = \frac{p}{q}(P_x(m, n), P_y(m, n), P_z(m, n))$$

where m, n as well as p, q are coprime integers with $q > 0$ dividing $2\,\mathrm{lcm}(A, B)Cz_0^2$, and

$$\begin{aligned} P_x(m, n) &= x_0Am^2 + 2y_0Bmn - x_0Bn^2, \\ P_y(m, n) &= -y_0Am^2 + 2x_0Amn + y_0Bn^2, \\ P_z(m, n) &= z_0Am^2 + z_0Bn^2. \end{aligned}$$

Finiteness

Let $\{U_n(P,Q)\}$ be a Lucas sequence with $Q \in \{\pm 1\}$. Theorem A implies that

$$\sum_{k=1}^{\infty} \frac{U_{k-1}}{x^k} = \frac{1}{x^2 - Px \pm 1}. \tag{3}$$

Hence from equations (1) and (3) it follows that $U_n = x^2 - Px \pm 1$. Finiteness follows from results by Nemes and Pethő and by Pethő. We provide two approaches to determine a finite set of possible values of x for which $U_n = x^2 - Px \pm 1$.

Elliptic curves

The first one is based on elliptic curves. It only works if one can determine the rank of the appropriate Mordell-Weil groups. The second method is based on an elementary reduction algorithm which yield finitely many quartic Thue equations to solve. Substituting $U_n = x^2 - Px \pm 1$ into the identity (2) yields a genus 1 curve

$$C_{(P,Q,n)}: \quad y^2 = (P^2 - 4Q)(x^2 - Px + Q)^2 + 4Q^n.$$

Bruin and Stoll described and algorithm the so-called two-cover descent, which can be used to prove that a given hyperelliptic curve has no rational points.

Elliptic curves

This algorithm is implemented in Magma, the procedure is called `TwoCoverDescent`. If it turns out that there are no rational points on the curves $y^2 = (P^2 - 4)(x^2 - Px + 1)^2 + 4$ and $y^2 = (P^2 + 4)(x^2 - Px - 1)^2 \pm 4$, then the equation (1) has no solution. If `TwoCoverDescent` yields that rational points may exist, but the procedure `Points` fails to find one, then we follow the second approach, solution via Thue equations that we consider later in the proof. Now we assume that we could determine points on curves for which `TwoCoverDescent` predicts existence of rational points. That means we are given elliptic curves in quartic model. Tzanakis provided a method to determine all integral points on quartic models, the algorithm is implemented in Magma as `IntegralQuarticPoints`.

Elementary reduction

There are three ternary quadratic equations to parametrize

$$\begin{aligned} Q_1: \quad X^2-(P^2-4)Y^2-4Z^2 &= 0,\\ Q_2: \quad X^2-(P^2+4)Y^2-4Z^2 &= 0,\\ Q_3: \quad X^2-(P^2+4)Y^2+4Z^2 &= 0. \end{aligned}$$

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Q_3: \quad X^2-(P^2+4)Y^2+4Z^2 &= 0.
\end{aligned}
$$

There are points on these curves:

$$
\begin{aligned}
Q_1: &\quad (X,Y,Z)=(2,0,1),\\
Q_2: &\quad (X,Y,Z)=(2,0,1),\\
Q_3: &\quad (X,Y,Z)=(P,1,1).
\end{aligned}
$$

Elementary reduction

It follows from Lemma 1 that
$\pm 1 = P_z(m,n) = \frac{p}{q}(m^2 - (P^2 \pm 4)n^2)$. Therefore $p = \pm 1$. We deal with the curve $y^2 = (P^2-4)(x^2 - Px + 1)^2 + 4$, the other two cases are similar. We obtain that

$$x^2 - Px + 1 = \frac{\pm 4mn}{q}.$$

Hence we have that

$$q(2x-P)^2 \pm (4-P^2)(m^2 - (P^2-4)n^2) \mp 16mn = 0,$$

where $q > 0$ divides $8(P^2-4)$. Applying Lemma 1 again we obtain that $m = f_m(u,v)$ and $n = f_n(u,v)$, where f_m, f_n are homogeneous quadratic polynomials.

Eigenvalues and eigenvectors

If we have a possible solution $x \in \mathbb{N}$ of equation (1), then we have to compute the value of the sum $\sum_{k=1}^{\infty} \frac{U_{k-1}}{x^k}$. We define

$$T = \begin{pmatrix} P/x & -Q/x \\ 1/x & 0 \end{pmatrix}.$$

Following standard arguments one has that

$$\frac{1}{x}\left(T^0 + T^1 + T^2 + \ldots + T^{N-1}\right)\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} * \\ \sum_{k=1}^{N} \frac{U_{k-1}}{x^k} \end{pmatrix}.$$

Using eigenvectors and eigenvalues one can determine a formula for the powers of T, hence one obtains a formula depending only on N for the sum $\sum_{k=1}^{N} \frac{U_{k-1}}{x^k}$.

Illustration of the algorithm

We will illustrate how one can use the approaches provided in the proof of Theorem 1 to determine all solutions of equation (1) for given P and Q. First we deal with the case $P = 4, Q = -1$. We have that

$$y^2 = 20(x^2 - 4x - 1)^2 \pm 4.$$

To determine all integral solutions we use the Magma commands

`IntegralQuarticPoints([20, -160, 280, 160, 16])` and
`IntegralQuarticPoints([20, -160, 280, 160, 24],[-1,-18])`.

One obtains that $x \in \{-643, -1, 0, 1, 3, 4, 5, 647\}$.

Illustration of the algorithm

Since $x \geq 2$ only 4 values remain. In case of $x = 647$ the matrix T is as follows

$$\begin{pmatrix} 4/647 & 1/647 \\ 1/647 & 0 \end{pmatrix}$$

and we obtain that

$$\sum_{k=1}^{N} \frac{U_{k-1}}{647^k} = \frac{\left(\frac{2-\sqrt{5}}{647}\right)^N (129\sqrt{5}-1) - \left(\frac{2+\sqrt{5}}{647}\right)^N (129\sqrt{5}+1)}{832040} + \frac{1}{416020}.$$

Nemzeti Kiválóság Program

Illustration of the algorithm

Thus

$$\lim_{N\to\infty}\sum_{k=1}^{N}\frac{U_{k-1}}{647^k}=\frac{1}{416020}=\frac{1}{U_{10}}.$$

In a similar way we get that

$$\begin{aligned}\lim_{N\to\infty}\sum_{k=1}^{N}\frac{U_{k-1}}{3^k} &= +\infty,\\ \lim_{N\to\infty}\sum_{k=1}^{N}\frac{U_{k-1}}{4^k} &= +\infty,\\ \lim_{N\to\infty}\sum_{k=1}^{N}\frac{U_{k-1}}{5^k} &= \frac{1}{4}=\frac{1}{U_2}.\end{aligned}$$

Illustration of the algorithm

We apply the second method to completely solve equation (1) with $P = 3, Q = 1$. The curve $C_{P,Q,n}$ has the form $y^2 = 5(x^2 - 3x + 1)^2 + 4$. It can be written as $v^2 = 5u^4 - 50u^2 + 189$ with $v = 4y$ and $u = 2x - 3$. The second approach has been implemented in Sage by Alekseyev and Tengely. Using their procedure `QuarticEq([5,-50,189])` we obtain that $u \in \{\pm 1, \pm 3, \pm 15\}$, therefore $x \in \{-6, 0, 1, 2, 3, 9\}$. We have that

$$\begin{aligned}
\lim_{N\to\infty} \sum_{k=1}^{N} \frac{U_{k-1}}{2^k} &= +\infty, \\
\lim_{N\to\infty} \sum_{k=1}^{N} \frac{U_{k-1}}{3^k} &= 1 = \frac{1}{U_1}, \\
\lim_{N\to\infty} \sum_{k=1}^{N} \frac{U_{k-1}}{9^k} &= \frac{1}{55} = \frac{1}{U_5}.
\end{aligned}$$