ON THE LUCAS SEQUENCE EQUATION $\frac{1}{U_n} = \sum_{k=1}^{\infty} \frac{U_{k-1}}{x^k}$

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ABSTRACT. In 1953 Stancliff noted an interesting property of the Fibonacci number $F_{11} = 89$. One has that
$$\frac{1}{89} = \frac{0}{10} + \frac{1}{10^2} + \frac{1}{10^3} + \frac{2}{10^4} + \frac{3}{10^5} + \frac{5}{10^6} + \dots,$$
where in the numerators the elements of the Fibonacci sequence appear. We provide methods to determine similar identities in case of Lucas sequences. As an example we prove that
$$\frac{1}{U_{10}} = \frac{1}{416020} = \sum_{k=0}^{\infty} \frac{U_k}{647^{k+1}},$$
where $U_0 = 0, U_1 = 1$ and $U_n = 4U_{n-1} + U_{n-2}, n \geq 2$.

1. INTRODUCTION

Stancliff [16] noted without proof an interesting property of the Fibonacci sequence F_n. One has that

$$\frac{1}{F_{11}} = \frac{1}{89} = 0.0112358\ldots = \sum_{k=0}^{\infty} \frac{F_k}{10^{k+1}}.$$

In 1980 Winans [23] investigated the related sums

$$\sum_{k=0}^{\infty} \frac{F_{\alpha k}}{10^{k+1}}$$

for certain values of α. In 1981 Hudson and Winans [8] provided a complete characterization of all decimal fractions that can be approximated by sums of the type

$$\frac{1}{F_\alpha} \sum_{k=1}^{n} \frac{F_{\alpha k}}{10^{l(k+1)}}, \quad \alpha, l \geq 1.$$

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Long [12] proved a general identity for binary recurrence sequences from which one obtains e.g.

$$\frac{1}{9899} = \sum_{k=0}^{\infty} \frac{F_k}{10^{2(k+1)}}, \quad \frac{1}{109} = \sum_{k=0}^{\infty} \frac{F_k}{(-10)^{k+1}}.$$

In the previous examples decimal fractions were studied, in case of different bases characterizations were obtained by Jia Sheng Lee [10] and by Köhler [9] and by Jin Zai Lee and Jia Sheng Lee [11]. Here we state a result by Köhler that we will use later in this article.

Theorem A. *Let A, B, a_0, a_1 be arbitrary complex numbers. Define the sequence $\{a_n\}$ by the recursion $a_{n+1} = Aa_n + Ba_{n-1}$. Then the formula*

$$\sum_{k=0}^{\infty} \frac{a_k}{x^{k+1}} = \frac{a_0 x - Aa_0 + a_1}{x^2 - Ax - B}$$

holds for all complex x such that $|x|$ is larger than the absolute values of the zeros of $x^2 - Ax - B$.

Let P and Q be non-zero relatively prime integers. The Lucas sequence $\{U_n(P, Q)\}$ is defined by

$$U_0 = 0, U_1 = 1 \text{ and } U_n = PU_{n-1} - QU_{n-2}, \text{ if } n \geq 2.$$

In this paper we deal with the determination of all integers $x \geq 2$ for which there exists an $n \geq 0$ such that

$$\frac{1}{U_n} = \sum_{k=1}^{\infty} \frac{U_{k-1}}{x^k}, \tag{1}$$

where U_n is a Lucas sequence with some given P and Q. In case of $P = 1, Q = -1$ one gets the Fibonacci sequence. De Weger [5] computed all $x \geq 2$ in case of the Fibonacci sequence, the solutions are as follows

$$\frac{1}{F_1} = \frac{1}{F_2} = \frac{1}{1} = \sum_{k=1}^{\infty} \frac{F_{k-1}}{2^k}, \qquad \frac{1}{F_5} = \frac{1}{5} = \sum_{k=1}^{\infty} \frac{F_{k-1}}{3^k},$$

$$\frac{1}{F_{10}} = \frac{1}{55} = \sum_{k=1}^{\infty} \frac{F_{k-1}}{8^k}, \qquad \frac{1}{F_{11}} = \frac{1}{89} = \sum_{k=1}^{\infty} \frac{F_{k-1}}{10^k}.$$

De Weger applied arguments of algebraic number theory and obtained two Thue equations, which were solved using Baker's method (see e.g. [2, 7, 15]). In the current work, we show how to reduce a search for integral $x \geq 2$ related to the equation $\frac{1}{U_n} = \sum_{k=1}^{\infty} \frac{U_{k-1}}{x^k}$ to elliptic Diophantine equations or to Thue equations following an elementary argument by Alekseyev and Tengely [1]. There exists a number of software implementations for determining integral points on elliptic curves, these procedures are based on a method developed by Stroeker and Tzanakis [18] and independently by Gebel, Pethő and Zimmer [6]. The elliptic logarithm method for determining all integer points on an elliptic curve has been applied to a variety of elliptic equations (see e.g. [19, 20, 21, 22]).

2. Main results

Theorem 1. *Let $\{U_n(P,Q)\}$ be a Lucas sequence with $Q \in \{\pm 1\}$ and $(P,Q) \notin \{(-2,1),(2,1)\}$. Then equation* (1) *possesses only finitely many solutions in n, x which can be effectively determined.*

The proof of Theorem 1 provides an algorithm to determine all solutions of equation (1). Following this algorithm we obtain numerical results.

Theorem 2. *Let $\{U_n(P,Q)\}$ be a Lucas sequence with $-10 \leq P \leq 10, Q \in \{\pm 1\}$ and $(P,Q) \neq (-2,1),(2,1)$. Then equation* (1) *has the following solutions*

$$\begin{aligned}
&(P,Q,n,x) \in \{(-3,1,5,6),(-1,-1,5,2),(-1,-1,11,9),(1,-1,1,2),\\
&(1,-1,2,2),(1,-1,5,3),(1,-1,10,8),(1,-1,11,10),(2,-1,2,3),\\
&(3,-1,2,4),(3,1,1,3),(3,1,5,9),(4,-1,2,5),(4,-1,10,647),\\
&(4,1,1,4),(5,-1,2,6),(5,1,1,5),(6,-1,2,7),(6,1,1,6),\\
&(7,-1,2,8),(7,1,1,7),(8,-1,2,9),(8,1,1,8),(9,-1,2,10),\\
&(9,1,1,9),(10,-1,2,11),(10,1,1,10)\}.
\end{aligned}$$

Remark. We note that the sequences with $(P,Q) \in \{(-2,1),(2,1)\}$ are degenerate sequences, that is the discriminant $D = P^2 - 4Q$ is zero. One has that $U_n(-2,1) = (-1)^{n+1}n$ and $U_n(2,1) = n$. In these cases there are infinitely many solutions of equation (1):

$$\begin{aligned}
\frac{1}{(x+1)^2} = \frac{1}{U_{(x+1)^2}(-2,1)} &= \sum_{k=1}^{\infty} \frac{U_{k-1}(-2,1)}{x^k}, \text{ if } x \text{ is even},\\
\frac{1}{(x-1)^2} = \frac{1}{U_{(x-1)^2}(2,1)} &= \sum_{k=1}^{\infty} \frac{U_{k-1}(2,1)}{x^k}.
\end{aligned}$$

3. Auxiliary results

The Lucas sequences $\{U_n(P,Q)\}$ and associated Lucas sequences $\{V_n(P,Q)\}$ are defined by the same linear recurrent relation with $P, Q \in$

$\mathbb{Z} \setminus \{0\}$ but different initial terms:

$$\begin{aligned} U_0 = 0, U_1 = 1 \text{ and } \quad & U_n = PU_{n-1} - QU_{n-2}, \text{ if } n \geq 2, \\ V_0 = 2, V_1 = P \text{ and } \quad & V_n = PV_{n-1} - QV_{n-2}, \text{ if } n \geq 2. \end{aligned}$$

Terms of Lucas sequences and associated Lucas sequences satisfy the following identity

$$V_n^2 - DU_n^2 = 4Q^n, \tag{2}$$

where $D = P^2 - 4Q$. To determine the appropriate Thue equations we use parametric solutions of ternary quadratic equations. Such parametrization are given in [1, Theorem 1].

Lemma 1. *Let A, B, C be non-zero integers and let (x_0, y_0, z_0) with $z_0 \neq 0$ be a particular non-trivial integer solution to the Diophantine equation $Ax^2 + By^2 + Cz^2 = 0$. Then its general integer solution is given by*

$$(x, y, z) = \frac{p}{q}(P_x(m, n), P_y(m, n), P_z(m, n))$$

where m, n as well as p, q are coprime integers with $q > 0$ dividing $2\operatorname{lcm}(A, B)Cz_0^2$, and

$$\begin{aligned} P_x(m, n) &= x_0Am^2 + 2y_0Bmn - x_0Bn^2, \\ P_y(m, n) &= -y_0Am^2 + 2x_0Amn + y_0Bn^2, \\ P_z(m, n) &= z_0Am^2 + z_0Bn^2. \end{aligned}$$

4. PROOFS

Proof of Theorem 1. Let $\{U_n(P, Q)\}$ be a Lucas sequence with $Q \in \{\pm 1\}$. Theorem A implies that

$$\sum_{k=1}^{\infty} \frac{U_{k-1}}{x^k} = \frac{1}{x^2 - Px \pm 1}. \tag{3}$$

Hence from equations (1) and (3) it follows that $U_n = x^2 - Px \pm 1$. Finiteness follows from results by Nemes and Pethő [13] and by Pethő [14]. We provide two approaches to determine a finite set of possible values of x for which $U_n = x^2 - Px \pm 1$. The first one is based on elliptic curves. It only works if one can determine the ranks of the appropriate Mordell-Weil groups. The second method is based on an elementary reduction algorithm which yield finitely many quartic Thue equations to solve. After computing the finite set of possible values of x we use eigenvalues and eigenvectors to compute the sum $\sum_{k=1}^{\infty} \frac{U_{k-1}}{x^k}$.

Substituting $U_n = x^2 - Px \pm 1$ into the identity (2) yields a genus 1 curve

$$C_{(P,Q,n)}: \quad y^2 = (P^2 - 4Q)(x^2 - Px + Q)^2 + 4Q^n.$$

Bruin and Stoll [4] described and algorithmized the so-called two-cover descent, which can be used to prove that a given hyperelliptic curve has no rational points. This algorithm is implemented in Magma [3], the procedure is called `TwoCoverDescent`. If it turns out that there are no rational points on the curves $y^2 = (P^2 - 4)(x^2 - Px + 1)^2 + 4$ and $y^2 = (P^2 + 4)(x^2 - Px - 1)^2 \pm 4$, then equation (1) has no solution. If `TwoCoverDescent` yields that rational points may exist, but the procedure `Points` fails to find one, then we follow the second approach, solution via Thue equations that we consider later in the proof. Now we assume that we could determine points on curves for which `TwoCoverDescent` predicts existence of rational points. That means we are given elliptic curves in quartic model. Tzanakis [21] provided a method to determine all integral points on quartic models, the algorithm is implemented in Magma as `IntegralQuarticPoints`.

The second approach is based on Lemma 1. There are three ternary quadratic equations to parametrize

$$\begin{aligned}
Q_1: &\quad X^2 - (P^2 - 4)Y^2 - 4Z^2 &= 0,\\
Q_2: &\quad X^2 - (P^2 + 4)Y^2 - 4Z^2 &= 0,\\
Q_3: &\quad X^2 - (P^2 + 4)Y^2 + 4Z^2 &= 0.
\end{aligned}$$

There are points on these curves:

$$\begin{aligned}
Q_1: &\qquad (X, Y, Z) = (2, 0, 1),\\
Q_2: &\qquad (X, Y, Z) = (2, 0, 1),\\
Q_3: &\qquad (X, Y, Z) = (P, 1, 1).
\end{aligned}$$

It follows from Lemma 1 that $\pm 1 = P_z(m, n) = \frac{p}{q}(m^2 - (P^2 \pm 4)n^2)$. Therefore $p = \pm 1$. We deal with the curve $y^2 = (P^2 - 4)(x^2 - Px + 1)^2 + 4$, the other two cases are similar. We obtain that

$$x^2 - Px + 1 = \frac{\pm 4mn}{q}.$$

Hence we have that

$$q(2x - P)^2 \pm (4 - P^2)(m^2 - (P^2 - 4)n^2) \mp 16mn = 0,$$

where $q > 0$ divides $8(P^2 - 4)$. Applying Lemma 1 again we obtain that $m = f_m(u, v)$ and $n = f_n(u, v)$, where f_m, f_n are homogeneous quadratic polynomials. It remains to compute integral solutions of finitely many quartic Thue equations (it may happen that the defining quartic polynomials are reducible, in those cases the resolution is simpler)

$$\pm q = f_m(u,v)^2 - (P^2-4)f_n(u,v)^2.$$

If we have a possible solution $x \in \mathbb{N}$ of equation (1), then we have to compute the value of the sum $\sum_{k=1}^{\infty} \frac{U_{k-1}}{x^k}$. We define

$$T = \begin{pmatrix} P/x & -Q/x \\ 1/x & 0 \end{pmatrix}.$$

Following standard arguments one has that

$$\frac{1}{x}\left(T^0 + T^1 + T^2 + \ldots + T^{N-1}\right)\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} * \\ \sum_{k=1}^{N} \frac{U_{k-1}}{x^k} \end{pmatrix}.$$

Using eigenvectors and eigenvalues one can determine a formula for the powers of T, hence one obtains a formula depending only on N for the sum $\sum_{k=1}^{N} \frac{U_{k-1}}{x^k}$. To find solutions of the equation (1) it remains to compute

$$\lim_{N\to\infty} \sum_{k=1}^{N} \frac{U_{k-1}}{x^k}.$$

$\square$

Proof of Theorem 2. We will illustrate how one can use the approaches provided in the proof of Theorem 1 to determine all solutions of equation (1) for given P and Q. First we deal with the case $P = 4, Q = -1$. We have that

$$y^2 = 20(x^2-4x-1)^2 \pm 4.$$

To determine all integral solutions we use the Magma commands

`IntegralQuarticPoints([20, -160, 280, 160, 16])` and
`IntegralQuarticPoints([20, -160, 280, 160, 24],[-1,-18])`.

One obtains that $x \in \{-643, -1, 0, 1, 3, 4, 5, 647\}$. Since $x \geq 2$ only 4 values remain. In case of $x = 647$ the matrix T is as follows

$$\begin{pmatrix} 4/647 & 1/647 \\ 1/647 & 0 \end{pmatrix}$$

and we obtain that

$$\sum_{k=1}^{N} \frac{U_{k-1}}{647^k} = \frac{\left(\frac{2-\sqrt{5}}{647}\right)^N (129\sqrt{5}-1) - \left(\frac{2+\sqrt{5}}{647}\right)^N (129\sqrt{5}+1)}{832040} + \frac{1}{416020}.$$

Thus

$$\lim_{N\to\infty}\sum_{k=1}^{N}\frac{U_{k-1}}{647^k}=\frac{1}{416020}=\frac{1}{U_{10}}.$$

In a similar way we get that

$$\begin{aligned}\lim_{N\to\infty}\sum_{k=1}^{N}\frac{U_{k-1}}{3^k} &= +\infty,\\ \lim_{N\to\infty}\sum_{k=1}^{N}\frac{U_{k-1}}{4^k} &= +\infty,\\ \lim_{N\to\infty}\sum_{k=1}^{N}\frac{U_{k-1}}{5^k} &= \frac{1}{4}=\frac{1}{U_2}.\end{aligned}$$

We apply the second method to completely solve equation (1) with $P=3,Q=1$. The curve $C_{P,Q,n}$ has the form $y^2=5(x^2-3x+1)^2+4$. It can be written as $v^2=5u^4-50u^2+189$ with $v=4y$ and $u=2x-3$. The second approach has been implemented in Sage [17] by Alekseyev and Tengely [1]. Using their procedure `QuarticEq([5,-50,189])` we obtain that $u\in\{\pm1,\pm3,\pm15\}$, therefore $x\in\{-6,0,1,2,3,9\}$. We have that

$$\begin{aligned}\lim_{N\to\infty}\sum_{k=1}^{N}\frac{U_{k-1}}{2^k} &= +\infty,\\ \lim_{N\to\infty}\sum_{k=1}^{N}\frac{U_{k-1}}{3^k} &= 1=\frac{1}{U_1},\\ \lim_{N\to\infty}\sum_{k=1}^{N}\frac{U_{k-1}}{9^k} &= \frac{1}{55}=\frac{1}{U_5}.\end{aligned}$$

□

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