INTEGRAL POINTS AND ARITHMETIC PROGRESSIONS ON HESSIAN CURVES AND HUFF CURVES

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ABSTRACT. In this paper we provide bounds for the size of the integral points on Hessian curves

$$H_d : \quad x^3 + y^3 - dxy + 1 = 0$$

where $d \in \mathbb{Z}$ is a parameter and we also study the integral points on Huff curves

$$H_{a,b} : \quad x(ay^2 - 1) = y(bx^2 - 1)$$

with $a, b \in \mathbb{Z}$. We also deal with integral points on these types of curves with x-coordinates forming arithmetic progressions.

1. INTRODUCTION

Siegel [29] in 1926 proved that the equation

$$y^2 = a_0x^n + a_1x^{n-1} + \ldots + a_n =: f(x)$$

has only a finite number of integer solutions if f has at least three simple roots. In 1929 Siegel [30] classified all irreducible algebraic curves over $\mathbb{Q}$ on which there are infinitely many integral points. These curves must be of genus 0 and have at most 2 infinite valuations. These results are ineffective, that is, their proofs do not provide any algorithm for finding the solutions. In the 1960's Baker [4, 6] gave explicit lower bounds for linear forms in logarithms of the form

$$\Lambda = \sum_{i=1}^{n} b_i \log \alpha_i \neq 0$$

where $b_i \in \mathbb{Z}$ for $i = 1, \ldots, n$ and $\alpha_1, \ldots, \alpha_n$ are algebraic numbers ($\neq 0, 1$), and $\log \alpha_i, \ldots, \log \alpha_n$ denote fixed determinations of the logarithms. Baker [5] used his fundamental inequalities concerning linear forms in logarithms to derive bounds for the solutions of the elliptic equation $y^2 = ax^3 + bx^2 + cx + d$. This bound were improved by several

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authors see e.g. [10, 20]. Baker and Coates [7] extended this result to general genus 1 curves. Lang proposed [22] proposed a different method to prove the finiteness of integral points on genus 1 curves. This method makes use of the group structure of the genus 1 curve. Stroeker and Tzanakis [32] and independently Gebel, Pethő and Zimmer [16] worked out an efficient algorithm based on this idea to determine all integral points on elliptic curves. The elliptic logarithm method for determining all integer points on an elliptic curve has been applied to a variety of elliptic equations (see e.g. [33, 34, 35, 36, 37]). The disadvantage of this approach is that there is no known algorithm to determine the rank of the so-called Mordell-Weil group of an elliptic curve, which is necessary to determine all integral points on the curve. There are other methods that can be used in certain cases to determine all integral solutions of genus 1 curves. Poulakis [27] provided an elementary algorithm to determine all integral solutions of equations of the form $y^2 = f(x)$, where $f(x)$ is quartic monic polynomial with integer coefficients. Using the theory of Pellian equations, Kedlaya [21] described a method to solve the system of equations

$$\begin{cases} x^2 - a_1 y^2 = b_1, \\ P(x, y) = z^2, \end{cases}$$

where P is a given integer polynomial.

An arithmetic progression on a curve

$$F(x, y) = 0,$$

is an arithmetic progression in either the x or y coordinates. One can pose the following natural question. What is the longest arithmetic progression in the x coordinates? In case of linear polynomials, Fermat claimed and Euler proved that four distinct squares cannot form an arithmetic progression. Allison [2] found an infinite family of quadratics containing an integral arithmetic progression of length eight. The curve is

$$y^2 = \frac{1}{2}(k^2 - l^2)x^2 - \frac{5}{2}(k^2 - l^2)x + (3k^2 - 2l^2),$$

and the arithmetic progression is as follows

$$(-1, 6k^2 - 5l^2), (0, 3k^2 - 2l^2), (1, k^2), (2, l^2), (3, l^2), (4, k^2), (5, 3k^2 - 2l^2), (6, 6k^2 - 5l^2).$$

Arithmetic progressions on Pellian equations $x^2 - dy^2 = m$ have been considered by many mathematicians. Dujella, Pethő and Tadić [15] proved that for any four-term arithmetic progression, except $\{0, 1, 2, 3\}$ and $\{-3, -2, -1, 0\}$, there exist infinitely many pairs (d, m) such that the terms of the given progression are y-components of solutions. Pethő and Ziegler [26] dealt with 5-term progressions on Pellian equations.

Aguirre, Dujella and Peral [1] constructed 6-term arithmetic progression on Pellian equations parametrized by points on elliptic curve having positive rank. Pethő and Ziegler posed several open problems. One of them is as follows: "Can one prove or disprove that there are d and m with $d > 0$ and not a perfect square such that $y = 1, 3, 5, 7, 9$ are in arithmetic progression on the curve $x^2 - dy^2 = m$?" Recenlty, González-Jiménez [17] answered the question: there is not m and d not a perfect square such that $y = 1, 3, 5, 7, 9$ are in arithmetic progression on the curve $x^2 - dy^2 = m$. He constructed the related diagonal genus 5 curve and he applied covering techniques and the so-called elliptic Chabauty's method. Bremner [11] provided an infinite family of elliptic curve of Weierstrass form with 8 points in arithmetic progression. González-Jiménez [17] showed that these arithmetic progressions cannot be extended to 9 points arithmetic progressions. Bremner, Silverman and Tzanakis [13] dealt with the congruent number curve $y^2 = x^3 - n^2x$, they considered integral arithmetic progressions. If F is a cubic polynomial, then the problem is to determine arithmetic progressions on elliptic curves. Bremner and Campbell [14] found distinct infinite families of elliptic curves, with arithmetic progression of length eight. Campbell [14] produced infinite families of quartic curves containing an arithmetic progression of length 9. Ulas [38] constructed an infinite family of quartics containing a progression of length 12. Restricting to quartics possessing central symmetry MacLeod [23] discovered four examples of length 14 progressions (e.g. $y^2 = -17x^4 + 3130x^2 + 8551, x = -13, -11, \ldots, 13.$) Alvarado [3] extended MacLeod's list by determining 11 more examples of length 14 progressions (e.g. $y^2 = 627x^4 - 87870x^2 + 3312859$) Moody [24] proved that there are infinitely many Edwards curves with 9 points in arithmetic progression. Bremner [12] and independently González-Jiménez [17, 18] proved using elliptic Chabauty's method that Moody's examples cannot be extended to longer arithmetic progressions. Moody [25] produced six infinite families of Huff curves having the property that each has rational points with x-coordinate $x = -4, -3, \ldots, 3, 4$. That is he obtained arithmetic progressions of length 9.

2. MAIN RESULTS

We consider the cases $d \in \{0, 1, 2, 3\}$ separately since the general bounds are not valid for these values. Here we use the Magma [9] procedure `SIntegralDesbovesPoints` to determine all integral points except when $d = 3$.

Lemma 1. *The sets of integral points on the curves H_d for $d \in \{0,1,2,3\}$ are as follows*

$$\begin{aligned} &\{(-1,0),(0,-1)\} \text{ if } d \in \{0,1,2\},\\ &\{(x,-1-x): \quad x \in \mathbb{Z}\} \cup \{(1,1)\} \text{ if } d = 3. \end{aligned}$$

Proof. If $d \in \{0,1,2\}$, then the procedure `SIntegralDesbovesPoints` provides the appropriate sets of integral solutions. If $d = 3$, then the curve H_3 is singular. We have that

$$x^3 + y^3 - 3xy + 1 = (x+y+1)(x^2 - xy + y^2 - x - y + 1).$$

The factor $x + y + 1$ yields the points $(x, -1-x)$, the second factor has only one integral solution $(1,1)$. □

Theorem 1. *Let (x,y) be an integral point on the curve H_d, where $d \in \mathbb{Z} \setminus \{0,1,2,3\}$. If $d \leq -1$, then*

$$\min(\alpha_2, \beta_1) \leq x \leq \max(\alpha_1, \beta_2).$$

If $d \geq 4$, then

$$\min(\alpha_1, \gamma_1) \leq x \leq \max(\alpha_2, \gamma_2),$$

where

$$\begin{aligned}
\alpha_1 &= \frac{1}{3}\left(2\,d^3 - 2d\sqrt{d^4 - 27\,d} - 27\right)^{\frac{1}{3}},\\
\alpha_2 &= \frac{1}{3}\left(2\,d^3 + 2d\sqrt{d^4 - 27\,d} - 27\right)^{\frac{1}{3}},\\
\beta_1 &= -\frac{1}{6}\,d - \frac{1}{18}\sqrt{-12\,d^3 - 27\,d^2 - 18\,d + 321} - \frac{1}{6},\\
\beta_2 &= -\frac{1}{6}\,d + \frac{1}{18}\sqrt{-12\,d^3 - 27\,d^2 - 18\,d + 321} - \frac{1}{6},\\
\gamma_1 &= -\frac{1}{6}\,d - \frac{1}{18}\sqrt{12\,d^3 - 27\,d^2 + 18\,d - 327} + \frac{1}{6},\\
\gamma_2 &= -\frac{1}{6}\,d + \frac{1}{18}\sqrt{12\,d^3 - 27\,d^2 + 18\,d - 327} + \frac{1}{6}.
\end{aligned}$$

We apply the above theorem to determine all non-trivial arithmetic progression in case of $-1000 \leq d \leq 1000, d \neq 3$. We note that if $d = -t^2$, then the points $(-t,-1),(0,-1),(t,1)$ are on the curve H_d, hence there exist arithmetic progressions of length 3.

Theorem 2. *Let $-1000 \leq d \leq 1000, d \neq 3, d \neq -t^2$ for $t \in \mathbb{N}$. The complete list of values of d for which there exist non-trivial arithmetic progressions are given by the following table.*

d	*integral points*	*length of AP*
-25	$(-19,27)$,$(-13,-9)$,$(-9,-13)$,$(-7,-2)$,$(-5,-1)$,$(-2,-7)$, $(-1,5)$,$(-1,0)$,$(-1,-5)$, $(0,-1)$,$(5,-1)$,$(27,-19)$	5
5	$(-1,0),(0,-1),(1,2),(2,1)$	4
41	$(-13,4),(-1,0),(0,-1),(2,9),(4,-13),(5,14),(9,2),(14,5)$	4

In the following theorem we characterize the integral points on the curve $H_{a,b}$.

Theorem 3. *The Diophantine equation*

$$H_{a,b}: \quad x(ay^2-1)=y(bx^2-1)$$

with $a,b,x,y\in\mathbb{Z}$ has the following solutions

$$\begin{aligned}
(a,b,x,y) &= (a,a,x,x) \text{ with } a,x\in\mathbb{Z},\\
(a,b,x,y) &= (1,1,-1,1),\\
(a,b,x,y) &= (1,1,1,-1),\\
(a,b,x,y) &= (-1,-1,-1,1),\\
(a,b,x,y) &= (-1,-1,1,-1),\\
(a,b,x,y) &= (a,2-a,-1,1) \text{ with } a\in\mathbb{Z},\\
(a,b,x,y) &= (a,2-a,1,-1) \text{ with } a\in\mathbb{Z}.
\end{aligned}$$

A direct consequence of the above theorem is as follows.

Corollary 1. *Let $(x_1,y_1),(x_2,y_2),(x_3,y_3)$ be solutions of the equation $H_{a,b}$ for some $a,b\in\mathbb{Z}$ such that (x_1,x_2,x_3) forms an arithmetic progression and at most one solution (x_i,y_i) satisfies the condition $x_i=y_i$. Then $(x_1,x_2,x_3)=(-3,-1,1),(-1,0,1),(1,0,-1)$ or $(1,-1,-3)$.*

3. Proof of the results

Proof of Theorem 1. The polynomial $H_d(x,y)=x^3+y^3-dxy+1$ satisfies Runge's condition [19, 28]. The highest degree part is given by $x^3+y^3=(x+y)(x^2-xy+y^2)$. We follow the algorithm described in [8] to provide bounds for the size of the integral solutions. Let

$$x=\frac{1}{t},\quad y=\frac{s}{t}.$$

We obtain that $H_d(\frac{1}{t},\frac{s}{t})=\frac{1}{t^3}(1+s^3-dst+t^3)$. We apply Hensel lifting

$$\begin{aligned}
1+s^3-dst+t^3 = \big((s+1)+a_1t+a_2t^2+\ldots\big)\times\\
\big((s^2-s+1)+(b_1s+c_1)t+(b_2s+c_2)t^2+\ldots\big).
\end{aligned}$$

It turns out that the factors are up to order 4 as follows

$$\begin{aligned}
g_1 &= s+1+\frac{d}{3}t+\left(\frac{1}{3}-\frac{1}{81}d^3\right)t^3+O(t^4),\\
g_2 &= s^2-s+1+\left(-\frac{1}{3}d-\frac{1}{3}ds\right)t+\frac{1}{9}d^2t^2+\left(\frac{2}{3}-\frac{2}{81}d^3-(\frac{1}{3}-\frac{1}{81}d^3)s\right)t^3+O(t^4).
\end{aligned}$$

We determine a polynomial that vanishes on the branches given by g_1. Let $P(x,y)=A_0+B_0x+(A_1+B_1x)y+(A_2+B_2x)y^2$. Define

$$p(t,s)=t^3P\left(\frac{1}{t},\frac{s}{t}\right).$$

We compute $p \mod g_1$ to get the following system of equations:

$$\begin{aligned}\frac{1}{3}A_2d-A_1+B_0 &= 0,\\ \frac{1}{9}A_2d^2-\frac{1}{3}A_1d+A_0 &= 0,\\ \frac{1}{81}A_2d^3+\frac{1}{3}A_2 &= 0.\end{aligned}$$

That is $P_1(x,y)=3x+3y+d$. We also determine a polynomial that vanishes on the branches given by g_2. Here we obtain that

$$P_2(x,y)=9(x^2-xy+y^2)-3d(x+y)+d^2.$$

We have that $P_i(x,y)\to 0$ as we move to infinity along one of the branches, that is $P_i(x,y)=0$ if y is "large". There are three possibilities. First we deal with the "small" solutions, that is $x\in\mathbb{Z}$ lie in between the smallest real root and the largest real root of the discriminant of H_d with respect to y. We have that

$$\operatorname{disc}_y(x^3+y^3-dxy+1)=-27x^6+(4d^3-54)x^3-27.$$

The real roots of this polynomial are

$$\begin{aligned}\alpha_1 &= \frac{1}{3}\left(2\,d^3-2d\sqrt{d^4-27\,d}-27\right)^{\frac{1}{3}},\\ \alpha_2 &= \frac{1}{3}\left(2\,d^3+2d\sqrt{d^4-27\,d}-27\right)^{\frac{1}{3}}.\end{aligned}$$

Now we consider the case of "large" solutions. We have that

$$\begin{aligned}\operatorname{Res}_y(H_d,P_1-1) &= -27x^2+(-9d+9)x+d^3-3d^2+3d-28,\\ \operatorname{Res}_y(H_d,P_1+1) &= 27x^2+(9d+9)x+d^3+3d^2+3d-26,\\ \operatorname{Res}_y(H_d,P_2-1) &= 27x^2+(-9d^3+9d+243)x+d^6-3d^4-54d^3+3d^2+81d+728,\\ \operatorname{Res}_y(H_d,P_2+1) &= 27x^2+(9d^3+9d-243)x+d^6+3d^4-54d^3+3d^2-81d+730.\end{aligned}$$

The real zeros of $\operatorname{Res}_y(H_d,P_1+1)$ are

$$\begin{aligned}\beta_1 &= -\frac{1}{6}\,d-\frac{1}{18}\sqrt{-12\,d^3-27\,d^2-18\,d+321}-\frac{1}{6},\\ \beta_2 &= -\frac{1}{6}\,d+\frac{1}{18}\sqrt{-12\,d^3-27\,d^2-18\,d+321}-\frac{1}{6}\end{aligned}$$

and the real zeros of $\mathrm{Res}_y(H_d, P_1 - 1)$ are

$$\begin{aligned}\gamma_1 &= -\frac{1}{6}\,d - \frac{1}{18}\sqrt{12\,d^3 - 27\,d^2 + 18\,d - 327} + \frac{1}{6},\\ \gamma_2 &= -\frac{1}{6}\,d + \frac{1}{18}\sqrt{12\,d^3 - 27\,d^2 + 18\,d - 327} + \frac{1}{6}.\end{aligned}$$

We note that the last two polynomials $\mathrm{Res}_y(H_d, P_2-1)$ and $\mathrm{Res}_y(H_d, P_2+1)$ have real roots only in cases $0 < d < 4$. Lemma 1 provides the list of integral points on these curves. If (x, y) is an integral point with

$$\min(\alpha_2, \beta_1) \leq x \leq \max(\alpha_1, \beta_2) \text{ when } d \leq -1$$

or

$$\min(\alpha_1, \gamma_1) \leq x \leq \max(\alpha_2, \gamma_2) \text{ when } d \geq 4,$$

then $|P_1(x, y)| < 1$ or $|P_2(x, y)| < 1$. In other words, we have $P_1(x, y) = 0$ or $P_2(x, y) = 0$ for such points. To determine these points we need to compute the integral solutions of $\mathrm{Res}_y(H_d, P_1)$ and $\mathrm{Res}_y(H_d, P_1)$. We have that

$$\begin{aligned}\mathrm{Res}_y(H_d, P_1) &= (d-3)(d^2+3d+9),\\ \mathrm{Res}_y(H_d, P_2) &= (d-3)^2(d^2+3d+9)^2.\end{aligned}$$

Therefore the only possible d is 3, which is handled in Lemma 1. □

Proof of Theorem 2. We implemented an algorithm in Sage [31] based on Theorem 1 to determine all integral points on H_d. □

Proof of Theorem 3. Consider the case $a = b$. We obtain that

$$axy(y - x) = x - y.$$

Therefore $x = y$ is a solution for all $x \in \mathbb{Z}$. Assume that $x \neq y$. We get that $axy = -1$. Hence $(a, b, x, y) \in \{(-1, -1, \mp 1, \pm 1), (1, 1, \mp 1, \pm 1)\}$ are the possible solutions of the equation, and one can check that these are in fact solutions.

We may assume that $|a| > |b|$. We rewrite the equation in the form

$$byx^2 + (1 - ay^2)x - y = 0.$$

Thus there exists an integer t such that

$$F(y) := a^2y^4 + (4b - 2a)y^2 + 1 = t^2. \quad (1)$$

This equation satisfies Runge's condition so we apply Runge's method to determine all the integral solutions. Define $P(y) = ay^2 + \frac{2b-a}{a}$. We have that

$$\begin{aligned} F(y)-\left(P(y)-\frac{1}{a}\right)^2 &= 2y^2+\frac{4b}{a}-\frac{2}{a}-\frac{4b^2}{a^2}+\frac{4b}{a^2}-\frac{1}{a^2},\\ F(y)-\left(P(y)+\frac{1}{a}\right)^2 &= -2y^2+\frac{4b}{a}+\frac{2}{a}-\frac{4b^2}{a^2}-\frac{4b}{a^2}-\frac{1}{a^2}. \end{aligned}$$

These two quadratic polynomials have opposite signs if $|y| \geq 3$, since $|a| > |b|$. Therefore one has that

$$\left(P(y)-\frac{1}{a}\right)^2 < F(y) = t^2 < \left(P(y)+\frac{1}{a}\right)^2$$

if $|y| \geq 3$. It yields that $t = ay^2 + \frac{2b-a}{a}$. Equation (1) implies that $b = 0$. In this case

$$y \in \left\{\frac{-1}{2ax} \pm \sqrt{\frac{1}{4a^2x^2}+\frac{1}{a}}\right\}$$

and we obtain that $|y| \leq 1$. It remains to check the cases $y \in \{0, \pm 1, \pm 2\}$. One gets that $(x, y) = (\mp 1, \pm 1)$ and $b = 2 - a$. □

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