ON A GENERALIZATION OF A PROBLEM OF ERDŐS AND GRAHAM

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Dedicated to Professor Lajos Tamássy on his 90th birthday

Abstract. In this paper we provide bounds for the size of the solutions of the Diophantine equation $\frac{x(x+1)(x+2)(x+3)}{(x+a)(x+b)} = y^2$, where $a, b \in \mathbb{Z}, a \neq b$ are parameters. We also determine all integral solutions for $a, b \in \{-4, -3, -2, -1, 4, 5, 6, 7\}$.

1. introduction

Let us define

$$f(x, k, d) = x(x + d) \cdots (x + (k - 1)d).$$

Erdős [12] and independently Rigge [26] proved that if $x \geq 1$ and $k \geq 2$, then $f(x, k, 1)$ is never a perfect square. A celebrated result of Erdős and Selfridge [13] states that $f(x, k, 1)$ is never a perfect power of an integer, provided $x \geq 1$ and $k \geq 2$. That is, they completely solved the Diophantine equation

$$f(x, k, d) = y^l \tag{1}$$

with $d = 1$. The literature of this type of Diophantine equations is very rich. First consider some results related to $l = 2$. Euler proved (see [10] pp. 440 and 635) that a product of four terms in arithmetic progression is never a square solving (1) with $k = 4, l = 2$. Obláth [25] obtained a similar statement for $k = 5$. Saradha and Shorey [30] proved that (1) has no solutions with $k \geq 4$, provided that d is a power of a prime number. Laishram and Shorey [23] extended this result to the case where either $d \leq 10^{10}$, or d has at most six prime divisors. Bennett, Bruin, Győry and Hajdu [3] solved (1) with $6 \leq k \leq 11$ and $l = 2$. Hirata-Kohno, Laishram, Shorey and Tijdeman [22] completely solved (1) with $3 \leq k < 110$.

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Now assume for this paragraph that $l \geq 3$. Many authors have considered the more general equation

$$f(x, k, d) = by^l, \tag{2}$$

where $b > 0$ and the greatest prime factor of b does not exceed k. Saradha [29] proved that (2) has no solution with $k \geq 4$. Győry [16] studied the cases $k = 2, 3$, he determined all integral solutions. Győry, Hajdu and Saradha [18] proved that the product of four or five consecutive terms of an arithmetical progression of integers cannot be a perfect power, provided that the initial term is coprime to the difference. Hajdu, Tengely and Tijdeman [20] proved that the product of k coprime integers in arithmetic progression cannot be a cube when $2 < k < 39$. Hajdu and Kovács [19] proved that the product of k consecutive terms of a primitive arithmetic progression is never a fifth power when $3 \leq k \leq 54$. Győry, Hajdu and Pintér [17] proved that for any positive integers x, d and k with $\gcd(x, d) = 1$ and $3 < k < 35$, the product $x(x + d) \cdots (x + (k - 1)d)$ cannot be a perfect power.

Erdős and Graham [11] asked if the Diophantine equation

$$\prod_{i=1}^{r} f(x_i, k_i, 1) = y^2$$

has, for fixed $r \geq 1$ and $\{k_1, k_2, \ldots, k_r\}$ with $k_i \geq 4$ for $i = 1, 2, \ldots, r$, at most finitely many solutions in positive integers $(x_1, x_2, \ldots, x_r, y)$ with $x_i + k_i \leq x_{i+1}$ for $1 \leq i \leq r - 1$. Skałba [32] provided a bound for the smallest solution and estimated the number of solutions below a given bound. Ulas [35] answered the above question of Erdős and Graham in the negative when either $r = k_i = 4$, or $r \geq 6$ and $k_i = 4$. Bauer and Bennett [2] extended this result to the cases $r = 3$ and $r = 5$. Bennett and Van Luijk [4] constructed an infinite family of $r \geq 5$ non-overlapping blocks of five consecutive integers such that their product is always a perfect square. Luca and Walsh [24] studied the case $(r, k_i) = (2, 4)$ for all $i = 1, \ldots, r$.

In this paper we study the Diophantine equation

$$\frac{x(x+1)(x+2)(x+3)}{(x+a)(x+b)} = y^2, \tag{3}$$

where $a, b \in \mathbb{Z}, a \neq b$ are parameters. We provide bounds for the size of solutions and an algorithm to determine all solutions $(x, y) \in \mathbb{Z}^2$. The method of proof is based on Runge's method [15, 21, 27, 28, 31, 34, 37]. In 2008, Sankaranarayanan and Saradha established improved upper bounds for the size of the solutions of the Diophantine equations $F(x) = y^m$ and $F(x) = G(y)$, for which Runge's method can be applied.

They generalized the method to obtain bounds for the solutions of equations of the form $P(x)/Q(x) = y^m$. Based on this latter result we provide bounds for the solutions of equation (3). We note that solutions of (3) in integers also correspond to integer solutions to the hyperelliptic equation

$$x(x+1)(x+2)(x+3)(x+a)(x+b) = Y^2,$$

where $Y = (x+a)(x+b)y$. Baker [1] applied his theory of lower bounds for linear forms in logarithms to obtain upper bound for the size of solutions of hyperelliptic equations. Many authors improved the bound see e.g. [5, 7, 8, 9, 33, 36]. Still these bounds remain astronomical. It is also possible to apply Runge's method to provide upper bound for the size of integral solutions of this hyperelliptic curve. Our method yields better bound, thus it is more efficient to determine all integral solutions.

Theorem 1. *(I) If* $(x, y) \in \mathbb{Z}^2$ *is a solution of* (3) *with* $a \equiv b \pmod 2$, *then*

$$|x| \le \max\{|A_2|, |A_1|^{1/2}, |A_0|^{1/3}, |B_2|, |B_1|^{1/2}, |B_0|^{1/3}, |\frac{1}{4}(a+b-6)^2 ab|\},$$

where

$$\begin{aligned}
A_2 &= \frac{3}{4}a^2 + \frac{1}{2}ab + \frac{3}{4}b^2 - 2a - 2b + 7\\
A_1 &= -\frac{1}{4}a^3 + \frac{1}{4}a^2 b + \frac{1}{4}ab^2 + 2a^2 - \frac{1}{4}b^3 + 2b^2 - 4a - 4b + 6\\
A_0 &= -\frac{1}{4}(a+b-4)^2 ab\\
B_2 &= \frac{3}{4}a^2 + \frac{1}{2}ab + \frac{3}{4}b^2 - 4a - 4b - 5\\
B_1 &= -\frac{1}{4}a^3 + \frac{1}{4}a^2 b + \frac{1}{4}ab^2 + 4a^2 - \frac{1}{4}b^3 + 4b^2 - 16a - 16b + 6\\
B_0 &= -\frac{1}{4}(a+b-8)^2 ab.
\end{aligned}$$

(II) If $(x, y) \in \mathbb{Z}^2$ *is a solution of* (3) *with* $a \not\equiv b \pmod 2$, *then*

$$|x| \le 2\max\{|C_2|, |C_1|^{1/2}, |C_0|^{1/3}, |D_2|, |D_1|^{1/2}, |D_0|^{1/3}\},$$

where

$$\begin{aligned}
C_2 &= \frac{3}{4}a^2 + \frac{1}{2}ab + \frac{3}{4}b^2 - \frac{7}{2}a - \frac{7}{2}b - \frac{5}{4}\\
C_1 &= -\frac{1}{4}a^3 + \frac{1}{4}a^2b + \frac{1}{4}ab^2 + \frac{7}{2}a^2 - \frac{1}{4}b^3 + \frac{7}{2}b^2 - \frac{49}{4}a - \frac{49}{4}b + 6\\
C_0 &= -\frac{1}{4}(a+b-7)^2ab\\
D_2 &= \frac{3}{4}a^2 + \frac{1}{2}ab + \frac{3}{4}b^2 - \frac{5}{2}a - \frac{5}{2}b + \frac{19}{4}\\
D_1 &= -\frac{1}{4}a^3 + \frac{1}{4}a^2b + \frac{1}{4}ab^2 + \frac{5}{2}a^2 - \frac{1}{4}b^3 + \frac{5}{2}b^2 - \frac{25}{4}a - \frac{25}{4}b + 6\\
D_0 &= -\frac{1}{4}(a+b-5)^2ab.
\end{aligned}$$

We apply the above theorem to determine all integral solutions of (3) with $a, b \in \{-4, -3, -2, -1, 4, 5, 6, 7\}, a \neq b$.

Corollary 1. *All solutions* $(x, y) \in \mathbb{Z}^2, y \neq 0$ *of* (3) *with* $a, b \in \{-4, -3, -2, -1, 4, 5, 6, 7\}, a \neq b$ *are as follows*

$$\begin{aligned}
a = -4, b = -3, &\quad (x, y) \in \{(-6, 2), (1, 2)\}\\
a = -4, b = 5, &\quad (x, y) \in \{(-6, 6)\}\\
a = -2, b = 7, &\quad (x, y) \in \{(3, 6)\}\\
a = 6, b = 7, &\quad (x, y) \in \{(-4, 2), (3, 2)\}.
\end{aligned}$$

2. Proof of the results

In the proof we will use the following result of Fujiwara [14].

Lemma 1. *Put* $p(z) = \sum_{i=0}^{n} a_i z^i, a_n \neq 0,$ *where* $a_i \in \mathbb{R}$ *for all* $i = 0, 1, \ldots, n$. *Then*

$$\max\{|\zeta| : p(\zeta) = 0\} \leq 2 \max\left\{\left|\frac{a_{n-1}}{a_n}\right|, \left|\frac{a_{n-2}}{a_n}\right|^{1/2}, \ldots, \left|\frac{a_0}{a_n}\right|^{1/n}\right\}.$$

Proof of Theorem 1. The polynomial part of the Puiseux expansion of

$$\left(\frac{x(x+1)(x+2)(x+3)}{(x+a)(x+b)}\right)^{1/2}$$

is $x + 3 - \frac{a+b}{2}$.

(I) First we deal with the case $a \equiv b \pmod 2$ that is, when $\frac{a+b}{2}$ is an integer. We have that

$$x(x+1)(x+2)(x+3)-(x+a)(x+b)\left(x+2-\frac{a+b}{2}\right)^2 =$$
$$2x^3+A_2x^2+A_1x+A_0 =: f_A(x)$$

and

$$x(x+1)(x+2)(x+3)-(x+a)(x+b)\left(x+4-\frac{a+b}{2}\right)^2 =$$
$$-2x^3+B_2x^2+B_1x+B_0 =: f_B(x).$$

If follows from Lemma 1 that $f_A(x)\neq 0$ if

$$|x| > \max\{|A_2|, |A_1|^{1/2}, |A_0|^{1/3}\} =: r_A.$$

Similarly, one has that $f_B(x)\neq 0$ if

$$|x| > \max\{|B_2|, |B_1|^{1/2}, |B_0|^{1/3}\} =: r_B.$$

Therefore $f_A(x)f_B(x) < 0$, if $|x| > \max\{r_A, r_B\}$. We obtain that either

$$\left(x+4-\frac{a+b}{2}\right)^2 < \frac{x(x+1)(x+2)(x+3)}{(x+a)(x+b)} < \left(x+2-\frac{a+b}{2}\right)^2$$

or

$$\left(x+2-\frac{a+b}{2}\right)^2 < \frac{x(x+1)(x+2)(x+3)}{(x+a)(x+b)} < \left(x+4-\frac{a+b}{2}\right)^2.$$

Since $\frac{x(x+1)(x+2)(x+3)}{(x+a)(x+b)} = y^2$, we get that $y^2 = \left(x+3-\frac{a+b}{2}\right)^2$ in both cases. Thus x is a root of the quadratic polynomial $x(x+1)(x+2)(x+3)-(x+a)(x+b)\left(x+3-\frac{a+b}{2}\right)^2$. The constant term of this quadratic polynomial is $-\frac{1}{4}(a+b-6)^2ab$, hence

$$|x| \leq |\frac{1}{4}(a+b-6)^2ab|.$$

(II) Now we consider the case $a \not\equiv b \pmod 2$. We have that

$$x(x+1)(x+2)(x+3)-(x+a)(x+b)\left(x+3-\frac{a+b-1}{2}\right)^2 =$$
$$-x^3+C_2x^2+C_1x+C_0 =: f_C(x)$$

and

$$x(x+1)(x+2)(x+3)-(x+a)(x+b)\left(x+3-\frac{a+b+1}{2}\right)^2 =$$
$$x^3+D_2x^2+D_1x+D_0 =: f_D(x).$$

Lemma 1 implies that $f_C(x) \neq 0$ if

$$|x| > 2\max\{|C_2|, |C_1|^{1/2}, |C_0|^{1/3}\} =: r_C$$

and $f_D(x) \neq 0$ if

$$|x| > 2\max\{|D_2|, |D_1|^{1/2}, |D_0|^{1/3}\} =: r_D.$$

It is clear that $f_C(x)f_D(x) < 0$, if $|x| > \max\{r_C, r_D\}$. One gets that either

$$\left(x+3-\frac{a+b-1}{2}\right)^2 < \frac{x(x+1)(x+2)(x+3)}{(x+a)(x+b)} < \left(x+3-\frac{a+b+1}{2}\right)^2$$

or

$$\left(x+3-\frac{a+b+1}{2}\right)^2 < \frac{x(x+1)(x+2)(x+3)}{(x+a)(x+b)} < \left(x+3-\frac{a+b-1}{2}\right)^2.$$

In both cases we get a contradiction, since $\frac{x(x+1)(x+2)(x+3)}{(x+a)(x+b)} = y^2$ and there cannot be a square between consecutive squares. Thus $|x| \leq \max\{r_C, r_D\}$. □

Proof of Corollary 1. We wrote a Magma [6] code to solve equation (3). If $a \equiv b \pmod 2$, then we used the bound

$$|x| \leq \max\{|A_2|, |A_1|^{1/2}, |A_0|^{1/3}, |B_2|, |B_1|^{1/2}, |B_0|^{1/3}\}$$

and we determined the roots of the quadratic equation $x(x+1)(x+2)(x+3) - (x+a)(x+b)\left(x+3-\frac{a+b}{2}\right)^2$. Some details of the computations are given in the following table. We only indicate those cases where there is a solution with $y \neq 0$.

a	b	bound for $\|x\|$
-4	-3	96
-4	5	46
-2	7	50
6	7	114

□

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