ON A PROBLEM OF ERDŐS AND GRAHAM

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ABSTRACT. In this paper we provide bounds for the size of the solutions of the Diophantine equation

$$x(x+1)(x+2)(x+3)(x+k)(x+k+1)(x+k+2)(x+k+3) = y^2,$$

where $4 \leq k \in \mathbb{N}$ is a parameter. We also determine all integral solutions for $1 \leq k \leq 10^6$.

1. INTRODUCTION

Let us define

$$f(x,k,d) = x(x+d)\cdots(x+(k-1)d).$$

Erdős [6] and independently Rigge [17] proved that $f(x,k,1)$ is never a perfect square. A celebrated result of Erdős and Selfridge [7] states that $f(x,k,1)$ is never a perfect power of an integer, provided $x \geq 1$ and $k \geq 2$. That is, they completely solved the Diophantine equation

$$f(x,k,d) = y^l \tag{1}$$

with $d = 1$. The literature of this type of Diophantine equations is very rich. First consider some results related to $l = 2$. Euler proved (see [4] pp. 440 and 635) that a product of four terms in arithmetic progression is never a square solving (1) with $k = 4, l = 2$. Obláth [16] obtained a similar statement for $k = 5$. Saradha and Shorey [21] proved that (1) has no solutions with $k \geq 4$, provided that d is a power of a prime number. Laishram and Shorey [14] extended this result to the case where either $d \leq 10^{10}$, or d has at most six prime divisors. Bennett, Bruin, Győry and Hajdu [2] solved (1) with $6 \leq k \leq 11$ and $l = 2$. Hirata-Kohno, Laishram, Shorey and Tijdeman [13] completely solved (1) with $3 \leq k < 110$.

Now assume for this paragraph that $l \geq 3$. Many authors have considered the more general equation

$$f(x,k,d) = by^l, \tag{2}$$

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where $b > 0$ and the greatest prime factor of b does not exceed k. Saradha [20] proved that (2) has no solution with $k \geq 4$. Győry [9] studied the cases $k = 2, 3$, he determined all solutions. Győry, Hajdu and Saradha [10] proved that the product of four or five consecutive terms of an arithmetical progression of integers cannot be a perfect power, provided that the initial term is coprime to the difference. Hajdu, Tengely and Tijdeman [11] proved that the product of k coprime integers in arithmetic progression cannot be a cube when $2 < k < 39$. Győry, Hajdu and Pintér proved that for any positive integers x, d and k with $\gcd(x, d) = 1$ and $3 < k < 35$, the product $x(x+d)\cdots(x+(k-1)d)$ cannot be a perfect power.

Erdős and Graham [5] asked if the Diophantine equation

$$\prod_{i=1}^{r} f(x_i, k_i, 1) = y^2$$

has, for fixed $r \geq 1$ and $\{k_1, k_2, \ldots, k_r\}$ with $k_i \geq 4$ for $i = 1, 2, \ldots, r$, at most finitely many solutions in positive integers $(x_1, x_2, \ldots, x_r, y)$ with $x_i + k_i \leq x_{i+1}$ for $1 \leq i \leq r - 1$. Skałba [23] provided a bound for the smallest solution and estimated the number of solutions below a given bound. Ulas [27] answered the above question of Erdős and Graham in the negative when either $r = k_i = 4$, or $r \geq 6$ and $k_i = 4$. Bauer and Bennett [1] extended this result to the cases $r = 3$ and $r = 5$. Bennett and Van Luijk [3] constructed an infinite family of $r \geq 5$ non-overlapping blocks of five consecutive integers such that their product is always a perfect square. Luca and Walsh [15] studied the case $(r, k_i) = (2, 4)$.

In this paper we consider the Diophantine equation

$$x(x+1)(x+2)(x+3)(x+k)(x+k+1)(x+k+2)(x+k+3) = y^2, \tag{3}$$

where $4 \leq k \in \mathbb{N}$ is a parameter. We provide bounds for the size of solutions and an algorithm to determine all solutions $(x, y) \in \mathbb{N}^2$. The method of proof is based on Runge's method [8, 12, 18, 19, 22, 26, 28].

Theorem 1. *If $(x, y) \in \mathbb{N}^2$ is a solution of* (3) *then*

$$1 \leq x \leq 1.08k.$$

We apply the above theorem to determine all positive integral solutions of (3) with $4 \leq k \leq 10^6$.

Theorem 2. *The only solution $(x, y) \in \mathbb{N}^2$ of* (3) *with $4 \leq k \leq 10^6$ is*

$$(x, y) = (33, 3361826160)$$

with $k = 1647$.

2. PROOF OF THE RESULTS

Proof of Theorem 1. We apply Runge's method and we prove that large solutions do not exists and we provide bound for size of the possible small solutions. A solution to the equation (3) gives rise a solution to the equation

$$F(X) := X(X+k+2)(X+2k+2)(X+3k) = Y^2, \tag{4}$$

where $X = x^2+(k+3)x$. The polynomial part of the Puiseux expansion of $F(X)^{(1/2)}$ is

$$P(X) = X^2 + (3k+2)X + k^2 + 3k.$$

We obtain that

$$\begin{aligned} F(X) - (P(X)-1)^2 &= 2X^2 - (4k^2 - 6k + 4)X - k^4 - 6k^3 - 7k^2 + 6k - 1, \\ F(X) - (P(X)+1)^2 &= -2X^2 - (4k^2 + 6k + 4)X - k^4 - 6k^3 - 11k^2 - 6k - 1. \end{aligned}$$

Let α_1, α_2 be the roots of the quadratic polynomial $F(X)-(P(X)-1)^2$ and α_3, α_4 be the roots of $F(X)-(P(X)+1)^2$. We define $\beta_i, i = 1,2,3,4$ as follows

$$\beta_i = \begin{cases} |\alpha_i| & \text{if } \alpha_i \in \mathbb{R}, \\ 0 & \text{otherwise.} \end{cases}$$

It follows that

$$F(X) - (P(X)-1)^2 > 0, \quad \text{if } X \notin [\min_i\{\beta_i\}, \max_i\{\beta_i\}]$$

and

$$F(X) - (P(X)+1)^2 < 0, \quad \text{if } X \notin [\min_i\{\beta_i\}, \max_i\{\beta_i\}].$$

Hence we get that

$$(P(X)-1)^2 < F(X) < (P(X)+1)^2, \quad \text{if } X \notin [\min_i\{\beta_i\}, \max_i\{\beta_i\}].$$

If (X, Y) is a solution of (4) with $X \notin [\min_i\{\beta_i\}, \max_i\{\beta_i\}]$, then

$$Y = P(X).$$

It implies that

$$0 = F(X) - P(X)^2 = -4k^2X - k^4 - 6k^3 - 9k^2.$$

That is

$$X = -\left(\frac{k+3}{2}\right)^2.$$

Since $X = x^2 + (k+3)x$ we get that

$$x = \frac{-k-3}{2}.$$

It means that if there exists a large solution, then k has to be odd, $x = \frac{-k-3}{2}$ and $y = \frac{(k-3)(k-1)(k+1)(k+3)}{16}$. It is a contradiction since $k \geq 4$ and therefore $0 > \frac{-k-3}{2} = x$.

It remains to deal with the small solutions that is those with

$$X \in [\min_i\{\beta_i\}, \max_i\{\beta_i\}].$$

Hence we need to compute the roots of the polynomials $F(X)-(P(X)-1)^2$ and $F(X)-(P(X)+1)^2$. These are as follows

$$\begin{aligned}
\alpha_1 &= k^2 - \frac{3}{2}k - 1 - \frac{1}{2}\sqrt{6\,k^4 + 15\,k^2 + 6},\\
\alpha_2 &= k^2 - \frac{3}{2}k - 1 + \frac{1}{2}\sqrt{6\,k^4 + 15\,k^2 + 6},\\
\alpha_3 &= -k^2 - \frac{3}{2}k - 1 - \frac{1}{2}\sqrt{2\,k^4 - 5\,k^2 + 2},\\
\alpha_4 &= -k^2 - \frac{3}{2}k - 1 + \frac{1}{2}\sqrt{2\,k^4 - 5\,k^2 + 2}.
\end{aligned}$$

Since $k \geq 4$, we obtain that $6\,k^4 + 15\,k^2 + 6 \geq 0$ and $2\,k^4 - 5\,k^2 + 2 \geq 0$. Therefore $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbb{R}$ and we have

$$\alpha_3 < \alpha_4 < \alpha_1 < \alpha_2.$$

We need to solve the system of inequalities

$$\begin{aligned}
0 &\leq x^2 + (k+3)x - \alpha_3,\\
0 &\geq x^2 + (k+3)x - \alpha_2.
\end{aligned}$$

The first inequality is true for all $x \geq 1$. The second inequality implies that

$$-\frac{1}{2}k - \frac{1}{2}\sqrt{5\,k^2 + 2\sqrt{6\,k^4 + 15\,k^2 + 6} + 5} - \frac{3}{2} \leq x$$

and

$$x \leq -\frac{1}{2}k + \frac{1}{2}\sqrt{5\,k^2 + 2\sqrt{6\,k^4 + 15\,k^2 + 6} + 5} - \frac{3}{2}.$$

The lower bound is negative if $k > 0$, hence we have that $x > 0$, in case of the upper bound we obtain that $x \leq 1.08k$ if $k \geq 4$. $\square$

3. Algorithm to solve (3) for fixed k

Theorem 1 says that if there is a solution $(x, y) \in \mathbb{N}^2$ of the Diophantine equation (3), then $1 \leq x \leq 1.08k$. If k is small, then one can easily enumerate all solutions since the bound is linear in k. For larger values of k one can apply a sieve method similar to the Sieve of Eratosthenes, which eliminates composite numbers using small primes. There are many generalizations of the Sieve of Eratosthenes to solve different problems in number theory, cryptography (see e.g. [24] III.4.). We followed the steps described below to solve completely (3) in case of $4 \leq k \leq 10^6$.

(i) Define
$f(x) = x(x+1)(x+2)(x+3)(x+k)(x+k+1)(x+k+2)(x+k+3)$,
$F(p) = \{a : a \in [0 \dots p-1] \text{ and } f(a) \bmod p \text{ is a square in } \mathbb{F}_p\}$
and for a given interval I
$S_I = \{a : a \in I \cap \mathbb{N} \text{ such that } f(a) \text{ is a square}\}$.
(ii) If $4 \leq k \leq 1000$ one computes $S_{[1,1.08k]}$ by direct enumeration.
(iii) If $k > 1000$. Let $N = \log_{30} 1.08k$ and $M = \sqrt[N]{1.08k}$.
(iv) Let $p_1, p_2, \dots, p_{2N}$ be primes such that
$p_1 < p_2 < \dots < p_N \leq M < p_{N+1} < \dots < p_{2N}$.
(v) Compute $F(p_i)$ for all $i = 1, 2, \dots, 2N$.
(vi) Sort the sets $F(p_i)$ such that

$$\frac{|F(p_{i_j})|}{p_{i_j}} < \frac{|F(p_{i_{j+1}})|}{p_{i_{j+1}}}.$$

(vii) Using the Chinese remainder theorem determine $I = \{a : a \in [1, 1.08k] \cap \mathbb{N}, a \bmod p_{i_1} \in F(p_{i_1}), a \bmod p_{i_2} \in F(p_{i_2}), \dots, a \bmod p_{i_N} \in F(p_{i_N})\}$.
(viii) Compute S_I.

Note that here we used small primes around 30 having product about $1.08k$. For very small primes $|F(p)|/p$ is close to one since in this case for a given $a \in [0, \dots, p-1]$ we have that a is a root of $f(x)$. As an example consider the case $k = 1647$. Here we have $p_{i_1} = 47, p_{i_2} = 37$ and

$$\frac{|F(p_{i_1})|}{p_{i_1}} \approx 0.4468, \quad \frac{|F(p_{i_2})|}{p_{i_2}} \approx 0.5676.$$

Using the Chinese remainder theorem we obtain a set I having cardinality 441. So the cardinality of the search space is reduced by a factor about 4. We implemented the above algorithm in Sage [25]. The code can be downloaded from
`http://www.math.unideb.hu/~tengely/ErdosGraham.sage`.

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