On a Problem of Erdős and Graham

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Background

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- Many generalizations in the literature.
- Euler proved that a product of four terms in arithmetic progression is never a square.
- Obláth obtained a similar statement for $k = 5$.

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- Many nice results by Bruin, Bennett, Győry, Hajdu, Laishram, Pintér, Saradha, Shorey and others related to the Diophantine equation
- $f(x, k, d) = by^l$
- Techniques: Baker's method, modular approach, theory of elliptic curves, Chabauty's method, high degree Thue equations.

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Erdős and Graham asked if the Diophantine equation

$$\prod_{i=1}^{r} f(x_i, k_i, 1) = y^2$$

has, for fixed $r \geq 1$ and $\{k_1, k_2, \ldots, k_r\}$ with $k_i \geq 4$ for $i = 1, 2, \ldots, r$, at most finitely many solutions in positive integers $(x_1, x_2, \ldots, x_r, y)$ with $x_i + k_i \leq x_{i+1}$ for $1 \leq i \leq r - 1$.

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- Luca and Walsh (2007) studied the case $(r, k_i) = (2, 4)$.

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- Bauer and Bennett (2007) extended this result to the cases $r = 3$ and $r = 5$.
- Luca and Walsh (2007) studied the case $(r, k_i) = (2, 4)$.
- Bennett and Van Luijk (2012) constructed an infinite family of $r \geq 5$ non-overlapping blocks of five consecutive integers such that their product is always a perfect square.

Product of two blocks

We deal with the Diophantine equation

$$x(x+1)(x+2)(x+3)(x+k)(x+k+1)(x+k+2)(x+k+3) = y^2.$$

Theorem

If the above equation has a positive integer solution x, then

$$x < 2k - 2.$$

The only solution of the above Diophantine equation with $1 \leq x \leq 10^6$ is $(x, k, y) = (33, 1647, 3361826160)$.

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Product of two blocks

The equation

$$x(x+1)(x+2)(x+3)(x+k)(x+k+1)(x+k+2)(x+k+3)=y^2$$

can be rewritten as

$$X(X+k+2)(X+2k+2)(X+3k)=y^2,$$

where $X=x^2+(k+3)x$. Runge's method can be applied.

Application of Runge's method

Polynomial part of the Puiseux expansion:

$$X^2 + (3k+2)X + k^2 + 3k.$$

Define

$$\begin{aligned} F(X) &= X(X+k+2)(X+2k+2)(X+3k), \\ P_1(X) &= X^2 + (3k+2)X + k^2 + 3k - 1, \\ P_2(X) &= X^2 + (3k+2)X + k^2 + 3k + 1. \end{aligned}$$

Application of Runge's method

We have that

$$F(X) - P_1(X)^2 = 2X^2 - 2(2k^2 - 3k - 2)X - k^4 - 6k^3 - 7k^2 + 6k - 1,$$
$$F(X) - P_2(X)^2 = -2X^2 - 2(2k^2 + 3k + 2)X - k^4 - 6k^3 - 11k^2 - 6k - 1.$$

That is

$$F(X) - P_1(X)^2 > 0 \quad \text{if } X > C_1(k),$$
$$F(X) - P_2(X)^2 < 0 \quad \text{if } X > C_2(k).$$

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That is

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We obtain that

$$P_1(X)^2 < F(X) < P_2(X)^2$$

if X is large enough.

Bound for the solutions

We got that

$$P_1(X)^2 < F(X) < P_2(X)^2.$$

We also know that $F(X) = y^2$. Hence if X is large, then

$$y = X^2 + (3k+2)X + k^2 + 3k.$$

Therefore

$$0 = F(X) - y^2 = -4k^2X - k^4 - 6k^3 - 9k^2.$$

That is $X = -\left(\frac{k+3}{2}\right)^2$

Large solutions

It is easy to see that k has to be odd, so $k = 2t + 1$ and $X = -(t+2)^2$.
We also have that

$$x^2 + (2t+4)x = -(t+2)^2.$$

Hence we obtain that

$$x = -t - 2$$

and

$$(-t-2)(-t-1)(-t)(-t+1)(t-1)(t)(t+1)(t+2) = ((t-1)t(t+1)(t+2))^2.$$

Remark: it is a negative solution of the equation and we assumed that x is positive.

Small solutions

Fujiwara's result:

Lemma

Given $p(z) = \sum_{i=0}^{n} a_i z^i$, $a_n \neq 0$. *Then*

$$\max\{|\zeta| : p(\zeta) = 0\} \leq 2 \max \left\{ \left| \frac{a_{n-1}}{a_n} \right|, \left| \frac{a_{n-2}}{a_n} \right|^{1/2}, \ldots, \left| \frac{a_0}{a_n} \right|^{1/n} \right\}.$$

Application of Fujiwara's lemma:

$$F(X) - P_1(X)^2 \Rightarrow |X| \leq 2 \max \left\{ |2k^2 - 3k - 2|, \left| \frac{-k^4 - 6k^3 - 7k^2 + 6k - 1}{2} \right|^{1/2} \right\}$$

$$F(X) - P_2(X)^2 \Rightarrow |X| \leq 2 \max \left\{ |2k^2 + 3k + 2|, \left| \frac{-k^4 - 6k^3 - 11k^2 - 6k - 1}{2} \right|^{1/2} \right\}$$

Small solutions

We have

$$F(X) - P_1(X)^2 \Rightarrow |X| \leq 2\max\left\{|2k^2 - 3k - 2|, \left|\frac{k^2 + 3k - 1}{\sqrt{2}}\right|\right\}$$

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Upper bound for X is $4k^2 + 6k + 4$. That is

$$x^2 + (k+3)x < 4k^2 + 6k + 4.$$

An upper bound for x is $2k - 2$.

Elliptic curves

Certain estimates are valid if $k > 10 \Rightarrow$ genus 1 model:

$$X(X+k+2)(X+2k+2)(X+3k) = y^2.$$

Using the MAGMA procedure `IntegralQuarticPoints`$([1, 6k+4, 11k^2+18k+4, 6k^3+18k^2+12k, 0], [0, 0])$; it is possible to determine all integral points on these curves.

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k	$x^2+(k+3)x = X \in$
5	{-21,-16,-15,-12,-9,-7,3,0}
6	{-28,-18,-14,-12,-8,2,0}
7	{-36,-25,-21,-16,-12,-9,0}
8	{-45,-24,-18,-15,-10,0}
9	{-55,-36,-27,-20,-15,-11,1,0}
10	{-66,-30,-22,-18,-12,0}

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We only obtain solutions such that $x \leq 0$.

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- $S_1 = \{s : s(s+1)(s+2)(s+3)(s+k)(s+k+1)(s+k+2)(s+k+3)$ is a square in $\mathbb{F}_{p_1}\}$,
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- $SOL = \{CRT([a,b],[p_1,p_2]) : a \in S_1, b \in S_2\}$.

Examples

Let $k = 2013$. We have $p_1 = 67$ and $p_2 = 71$.
S_1 has 41 elements and S_2 has 39 elements.
SOL has 1360 elements less than or equal $2 \cdot 2013 - 2$. We obtain no solution for the original equation.

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Let $k = 2013 \cdot 9 \cdot 4 = 72468$. Here we use the primes $p_1 = 383$ and $p_2 = 389$.

$$|S_1| = 191 \quad |S_2| = 217 \quad |SOL| = 41447.$$

Improvement: $\sqrt[3]{2 \cdot 72468 - 2} \approx 41.7 \Rightarrow p_1 = 41, p_2 = 47, p_3 = 53,$
then we have

$$|SOL| = 12075.$$

Product of two blocks of length five

Consider the equation

$$F(x) = y^2,$$

where $F(x) = x(x+1)(x+2)(x+3)(x+4)(x+k)(x+k+1)(x+k+2)(x+k+3)(x+k+4)$. Polynomial part of the Puiseux expansion

$$\begin{aligned} P(x) \quad = \quad & x^5 + \left(\frac{5}{2}k + 10)x^4 + (\frac{15}{8}k^2 + 20k + 35\right) x^3 + \\ & \left(\frac{5}{16}k^3 + \frac{45}{4}k^2 + \frac{105}{2}k + 50\right) x^2 + \\ & \left(-\frac{5}{128}k^4 + \frac{5}{4}k^3 + \frac{145}{8}k^2 + 50k + 24\right) x + \\ & \frac{3}{256}k^5 - \frac{5}{64}k^4 + \frac{5}{16}k^3 + \frac{25}{4}k^2 + 12k. \end{aligned}$$

Application of Runge's method

We have

$$\begin{aligned} F(x) - \left(P(x) - \frac{1}{256}\right)^2 &> 0 \\ F(x) - \left(P(x) + \frac{1}{256}\right)^2 &< 0, \end{aligned}$$

if $x > C^+$. We have also

$$\begin{aligned} F(x) - \left(P(x) - \frac{1}{256}\right)^2 &< 0 \\ F(x) - \left(P(x) + \frac{1}{256}\right)^2 &> 0, \end{aligned}$$

if $x > C^-$.

Large solutions

We get that

$$\left(P(x)-\frac{1}{256}\right)^2 < F(x) < \left(P(x)+\frac{1}{256}\right)^2,$$

that is

$$(256P(x)-1)^2 < (256y)^2 < (256P(x)+1)^2.$$

It follows that $y = P(x)$ if $x > C^+$. If there is an integral solution x, then

$$x | k^2(k+4)^2(3k^3-32k^2+208k+768).$$

Other approach

Genus 2 model:

$$X(X+k+3)(X+2k+4)(X+3k+3)(X+4k)=y^2,$$

where $X=x^2+(k+4)x$. Computing integral points on genus 2 curves: Bugeaud, Mignotte, Siksek, Stoll and Tengely. One needs basis of Mordell-Weil group of the Jacobian.