

On a Problem of Erdős and Graham



Nemzeti
Kiválóság
Program

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21th Czech and Slovak
International Conference on
Number Theory
Ostravice

September 2-6, 2013

Background

Let us define

$$f(x, k, d) = x(x + d) \cdots (x + (k - 1)d).$$

- Erdős and independently Rigge proved that $f(x, k, 1)$ is never a perfect square.



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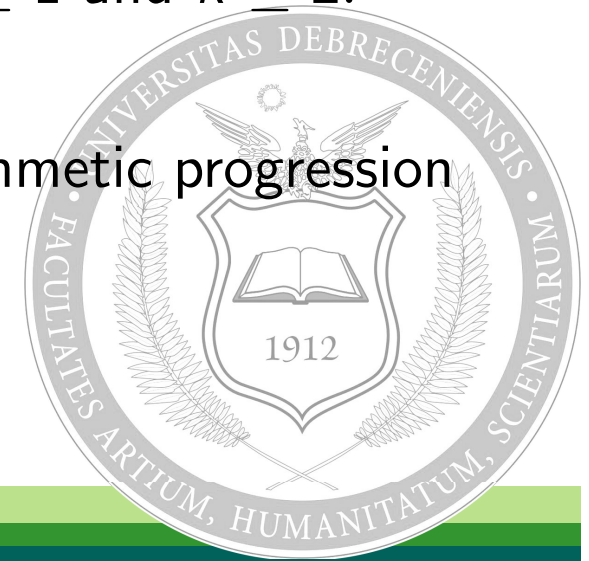


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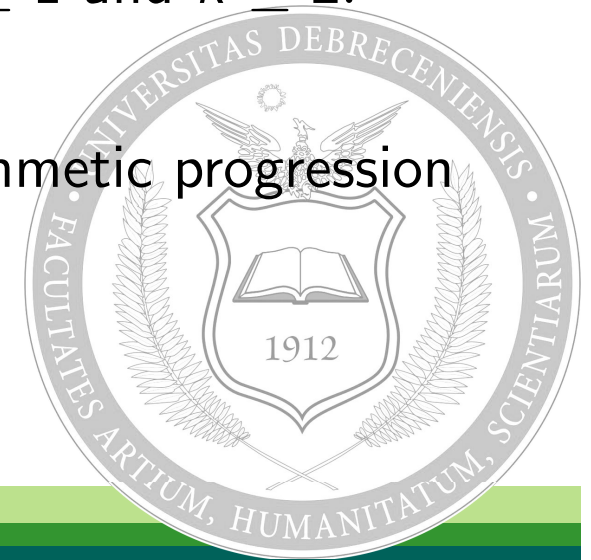


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- Obláth obtained a similar statement for $k = 5$.



Background

- Many nice results by Bruin, Bennett, Győry, Hajdu, Laishram, Pintér, Saradha, Shorey and others related to the Diophantine equation
- $f(x, k, d) = by^l$
- Techniques: Baker's method, modular approach, theory of elliptic curves, Chabauty's method, high degree Thue equations.



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Erdős and Graham asked if the Diophantine equation

$$\prod_{i=1}^r f(x_i, k_i, 1) = y^2$$

has, for fixed $r \geq 1$ and $\{k_1, k_2, \dots, k_r\}$ with $k_i \geq 4$ for $i = 1, 2, \dots, r$, at most finitely many solutions in positive integers $(x_1, x_2, \dots, x_r, y)$ with $x_i + k_i \leq x_{i+1}$ for $1 \leq i \leq r - 1$.



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- Luca and Walsh (2007) studied the case $(r, k_i) = (2, 4)$.
- Bennett and Van Luijk (2012) constructed an infinite family of $r \geq 5$ non-overlapping blocks of five consecutive integers such that their product is always a perfect square.



Product of two blocks

We deal with the Diophantine equation

$$x(x+1)(x+2)(x+3)(x+k)(x+k+1)(x+k+2)(x+k+3) = y^2.$$

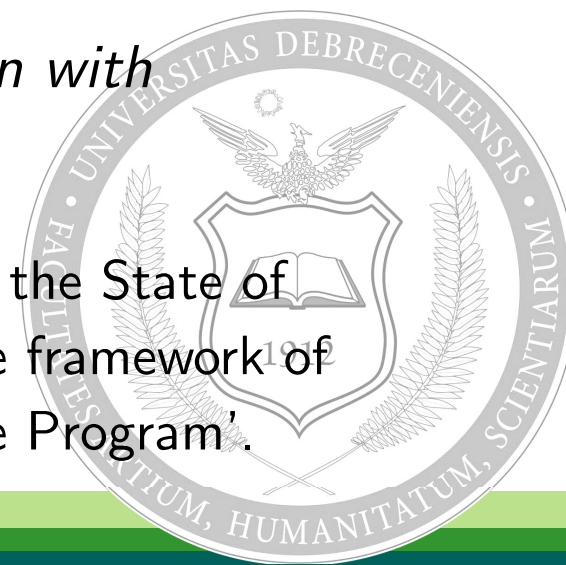
Theorem

If the above equation has a positive integer solution x , then

$$x < 2k - 2.$$

The only solution of the above Diophantine equation with $1 \leq x \leq 10^6$ is $(x, k, y) = (33, 1647, 3361826160)$.

This research was supported by the European Union and the State of Hungary, co-financed by the European Social Fund in the framework of TÁMOP 4.2.4. A/2-11-1-2012-0001 'National Excellence Program'.



Product of two blocks

The equation

$$x(x+1)(x+2)(x+3)(x+k)(x+k+1)(x+k+2)(x+k+3) = y^2$$

can be rewritten as

$$X(X+k+2)(X+2k+2)(X+3k) = y^2,$$

where $X = x^2 + (k+3)x$. Runge's method can be applied.



Application of Runge's method

Polynomial part of the Puiseux expansion:

$$X^2 + (3k + 2)X + k^2 + 3k.$$

Define

$$\begin{aligned} F(X) &= X(X + k + 2)(X + 2k + 2)(X + 3k), \\ P_1(X) &= X^2 + (3k + 2)X + k^2 + 3k - 1, \\ P_2(X) &= X^2 + (3k + 2)X + k^2 + 3k + 1. \end{aligned}$$



Application of Runge's method

We have that

$$F(X) - P_1(X)^2 = 2X^2 - 2(2k^2 - 3k - 2)X - k^4 - 6k^3 - 7k^2 + 6k - 1,$$

$$F(X) - P_2(X)^2 = -2X^2 - 2(2k^2 + 3k + 2)X - k^4 - 6k^3 - 11k^2 - 6k - 1.$$

That is

$$F(X) - P_1(X)^2 > 0 \quad \text{if } X > C_1(k),$$

$$F(X) - P_2(X)^2 < 0 \quad \text{if } X > C_2(k).$$



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That is

$$\begin{aligned}F(X) - P_1(X)^2 &> 0 && \text{if } X > C_1(k), \\F(X) - P_2(X)^2 &< 0 && \text{if } X > C_2(k).\end{aligned}$$

We obtain that

$$P_1(X)^2 < F(X) < P_2(X)^2$$

if X is large enough.



Bound for the solutions

We got that

$$P_1(X)^2 < F(X) < P_2(X)^2.$$

We also know that $F(X) = y^2$. Hence if X is large, then

$$y = X^2 + (3k + 2)X + k^2 + 3k.$$

Therefore

$$0 = F(X) - y^2 = -4k^2X - k^4 - 6k^3 - 9k^2.$$

That is $X = -\left(\frac{k+3}{2}\right)^2$



Large solutions

It is easy to see that k has to be odd, so $k = 2t + 1$ and $X = -(t + 2)^2$.

We also have that

$$x^2 + (2t + 4)x = -(t + 2)^2.$$

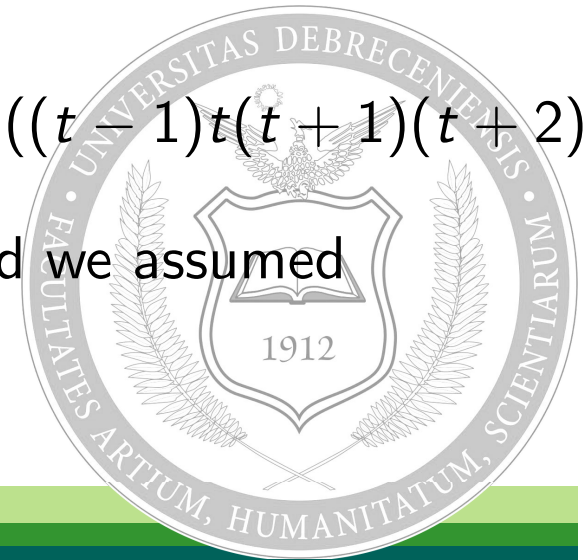
Hence we obtain that

$$x = -t - 2$$

and

$$(-t-2)(-t-1)(-t)(-t+1)(t-1)(t)(t+1)(t+2) = ((t-1)t(t+1)(t+2))^2.$$

Remark: it is a negative solution of the equation and we assumed that x is positive.



Small solutions

Fujiwara's result:

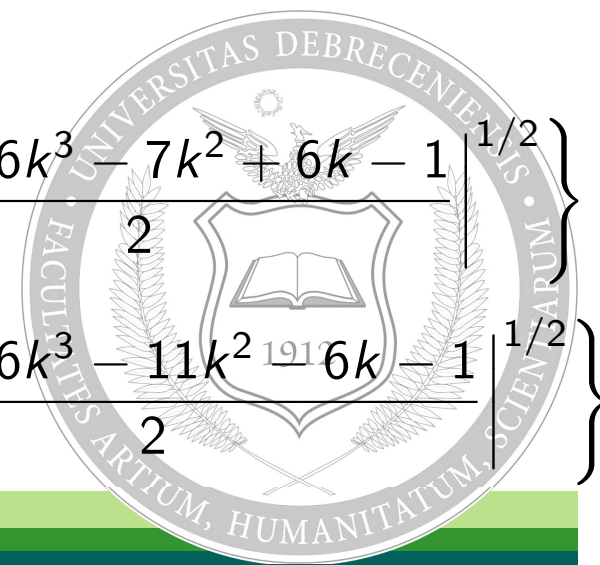
Lemma

Given $p(z) = \sum_{i=0}^n a_i z^i$, $a_n \neq 0$. Then

$$\max\{|\zeta| : p(\zeta) = 0\} \leq 2 \max \left\{ \left| \frac{a_{n-1}}{a_n} \right|, \left| \frac{a_{n-2}}{a_n} \right|^{1/2}, \dots, \left| \frac{a_0}{a_n} \right|^{1/n} \right\}.$$

Application of Fujiwara's lemma:

$$F(X) - P_1(X)^2 \Rightarrow |X| \leq 2 \max \left\{ |2k^2 - 3k - 2|, \left| \frac{-k^4 - 6k^3 - 7k^2 + 6k - 1}{2} \right|^{1/2} \right\}$$
$$F(X) - P_2(X)^2 \Rightarrow |X| \leq 2 \max \left\{ |2k^2 + 3k + 2|, \left| \frac{-k^4 - 6k^3 - 11k^2 + 6k - 1}{2} \right|^{1/2} \right\}$$



Small solutions

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$$F(X) - P_1(X)^2 \Rightarrow |X| \leq 2 \max \left\{ |2k^2 - 3k - 2|, \left| \frac{k^2 + 3k - 1}{\sqrt{2}} \right| \right\}$$

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Upper bound for X is $4k^2 + 6k + 4$. That is

$$x^2 + (k + 3)x < 4k^2 + 6k + 4.$$

An upper bound for x is $2k - 2$.



Elliptic curves

Certain estimates are valid if $k > 10 \Rightarrow$ genus 1 model:

$$X(X + k + 2)(X + 2k + 2)(X + 3k) = y^2.$$

Using the MAGMA procedure `IntegralQuarticPoints([1, 6k + 4, 11k2 + 18k + 4, 6k3 + 18k2 + 12k, 0], [0, 0])`; it is possible to determine all integral points on these curves.



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k	$x^2 + (k + 3)x = X \in$
5	$\{-21, -16, -15, -12, -9, -7, 3, 0\}$
6	$\{-28, -18, -14, -12, -8, 2, 0\}$
7	$\{-36, -25, -21, -16, -12, -9, 0\}$
8	$\{-45, -24, -18, -15, -10, 0\}$
9	$\{-55, -36, -27, -20, -15, -11, 1, 0\}$
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We only obtain solutions such that $x \leq 0$.

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- $S_1 = \{s : s(s + 1)(s + 2)(s + 3)(s + k)(s + k + 1)(s + k + 2)(s + k + 3) \text{ is a square in } \mathbb{F}_{p_1}\}$,
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- $SOL = \{CRT([a, b], [p_1, p_2]) : a \in S_1, b \in S_2\}$.



Examples

Let $k = 2013$. We have $p_1 = 67$ and $p_2 = 71$.

S_1 has 41 elements and S_2 has 39 elements.

SOL has 1360 elements less than or equal $2 \cdot 2013 - 2$. We obtain no solution for the original equation.



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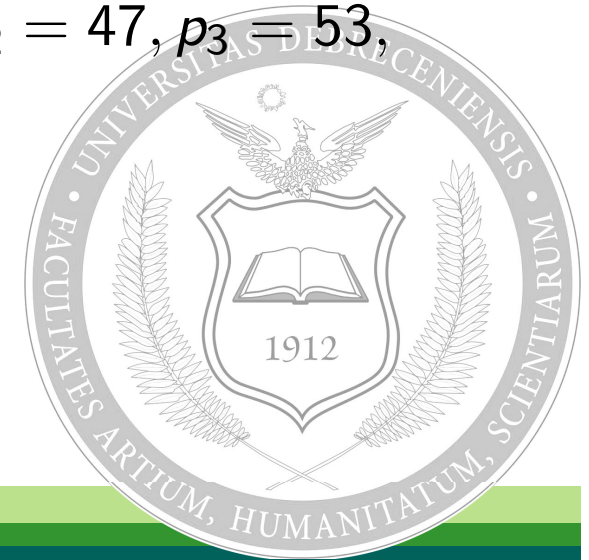
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Let $k = 2013 \cdot 9 \cdot 4 = 72468$. Here we use the primes $p_1 = 383$ and $p_2 = 389$.

$$|S_1| = 191 \quad |S_2| = 217 \quad |SOL| = 41447.$$

Improvement: $\sqrt[3]{2 \cdot 72468 - 2} \approx 41.7 \Rightarrow p_1 = 41, p_2 = 47, p_3 = 53$,
then we have

$$|SOL| = 12075.$$



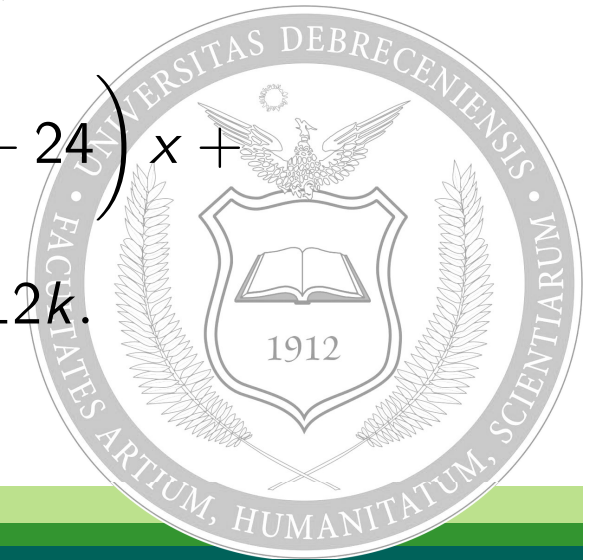
Product of two blocks of length five

Consider the equation

$$F(x) = y^2,$$

where $F(x) = x(x+1)(x+2)(x+3)(x+4)(x+k)(x+k+1)(x+k+2)(x+k+3)(x+k+4)$. Polynomial part of the Puiseux expansion

$$\begin{aligned} P(x) = & x^5 + \left(\frac{5}{2}k + 10 \right) x^4 + \left(\frac{15}{8}k^2 + 20k + 35 \right) x^3 + \\ & \left(\frac{5}{16}k^3 + \frac{45}{4}k^2 + \frac{105}{2}k + 50 \right) x^2 + \\ & \left(-\frac{5}{128}k^4 + \frac{5}{4}k^3 + \frac{145}{8}k^2 + 50k + 24 \right) x + \\ & \frac{3}{256}k^5 - \frac{5}{64}k^4 + \frac{5}{16}k^3 + \frac{25}{4}k^2 + 12k. \end{aligned}$$



Application of Runge's method

We have

$$F(x) - \left(P(x) - \frac{1}{256} \right)^2 > 0$$
$$F(x) - \left(P(x) + \frac{1}{256} \right)^2 < 0,$$

if $x > C^+$. We have also

$$F(x) - \left(P(x) - \frac{1}{256} \right)^2 < 0$$
$$F(x) - \left(P(x) + \frac{1}{256} \right)^2 > 0,$$

if $x > C^-$.



Large solutions

We get that

$$\left(P(x) - \frac{1}{256}\right)^2 < F(x) < \left(P(x) + \frac{1}{256}\right)^2,$$

that is

$$(256P(x) - 1)^2 < (256y)^2 < (256P(x) + 1)^2.$$

It follows that $y = P(x)$ if $x > C^+$. If there is an integral solution x , then

$$x | k^2(k + 4)^2(3k^3 - 32k^2 + 208k + 768).$$



Other approach

Genus 2 model:

$$X(X + k + 3)(X + 2k + 4)(X + 3k + 3)(X + 4k) = y^2,$$

where $X = x^2 + (k + 4)x$. Computing integral points on genus 2 curves: Bugeaud, Mignotte, Siksek, Stoll and Tengely. One needs basis of Mordell-Weil group of the Jacobian.

