Görbék és diofantikus egyenletek

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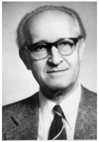
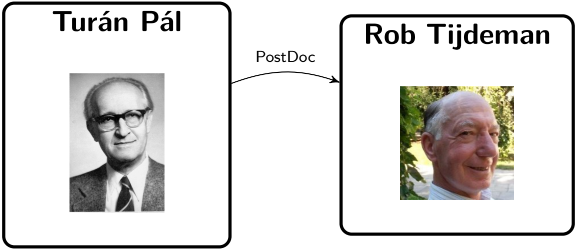
2017. május 23.

Turán Pál

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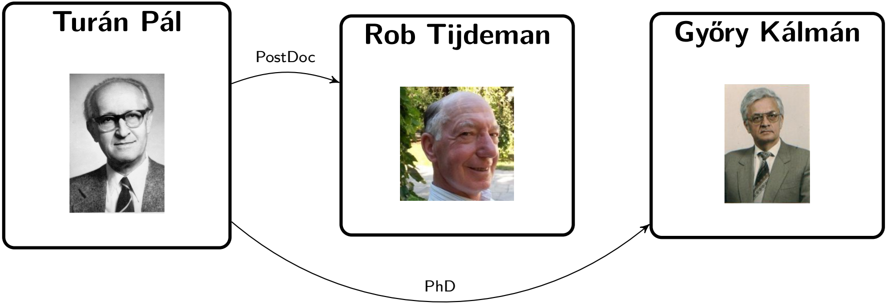
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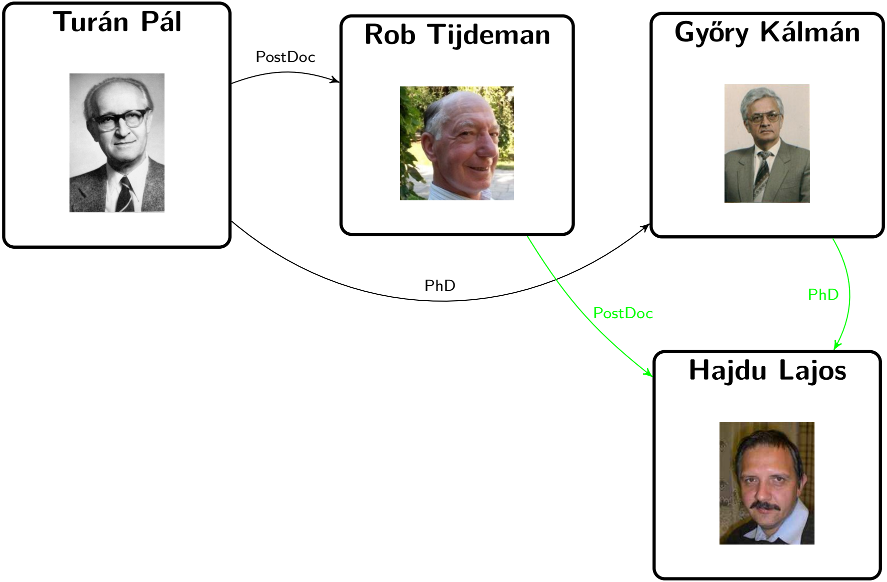
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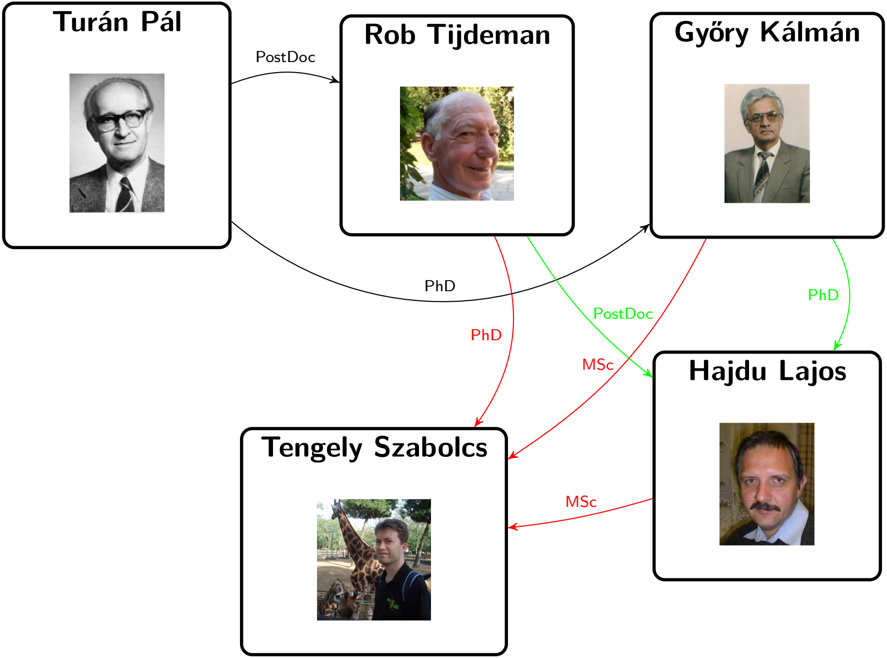
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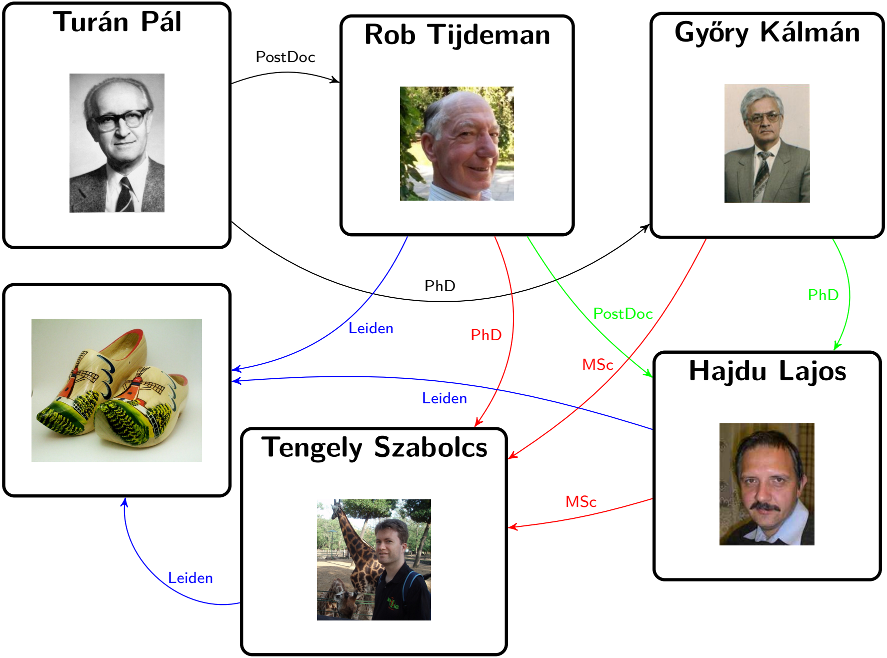
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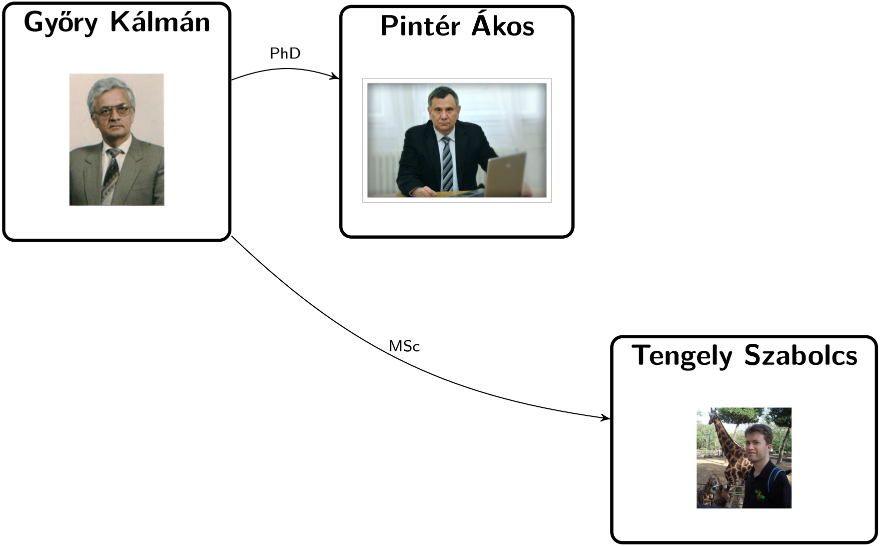
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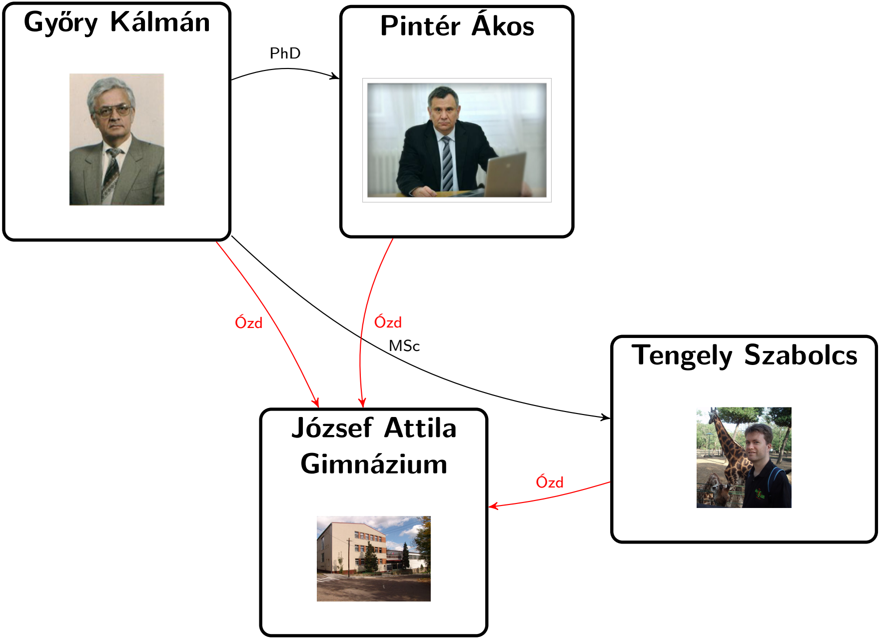
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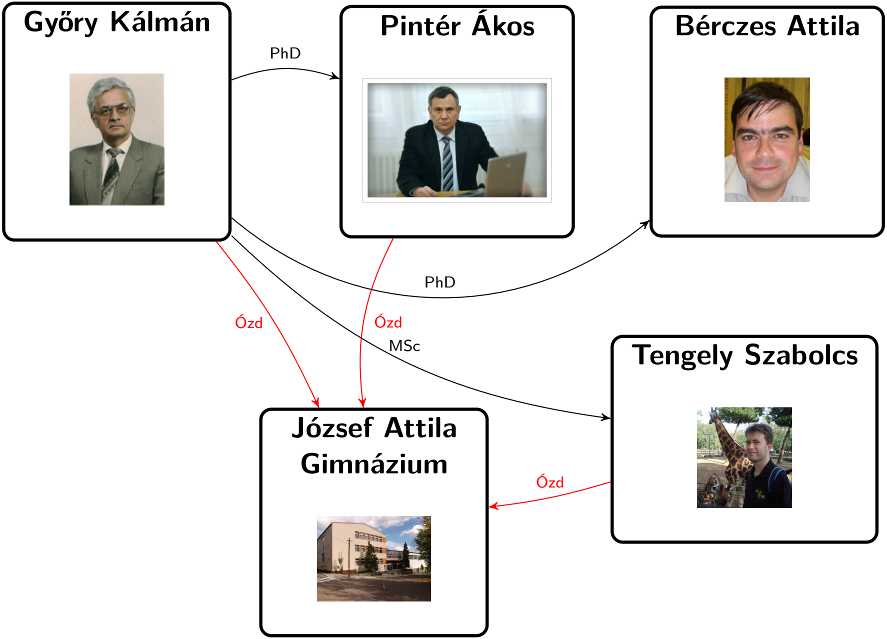

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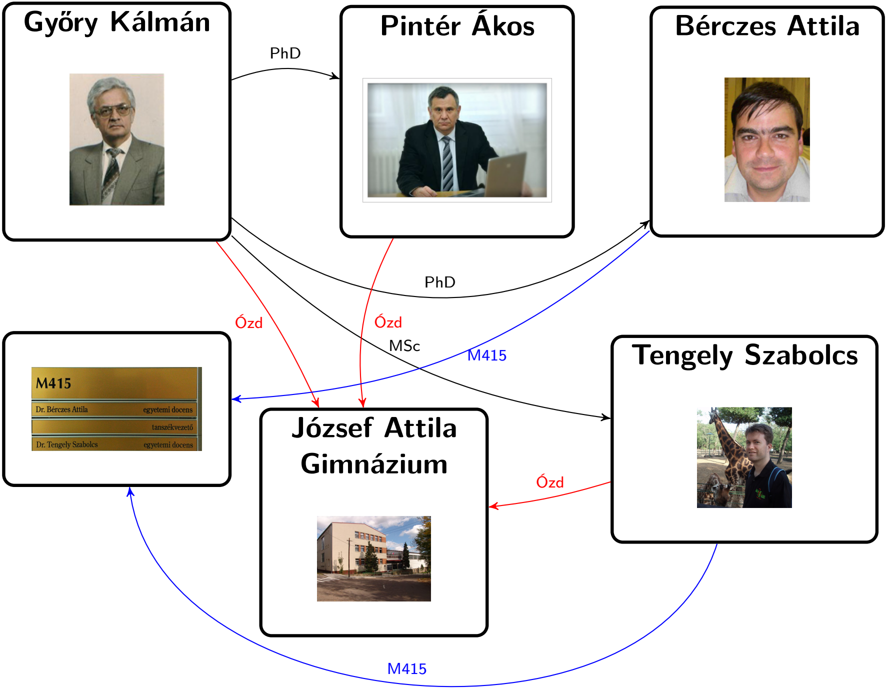
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Norm form equations and arithmetic progressions

Buchmann and Pethő found an interesting unit in the number field $K = \mathbb{Q}(\alpha)$ with $\alpha^7 - 3 = 0$ it is as follows

$$10 + 9\alpha + 8\alpha^2 + 7\alpha^3 + 6\alpha^4 + 5\alpha^5 + 4\alpha^6.$$

Norm form equations and arithmetic progressions

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$$10 + 9\alpha + 8\alpha^2 + 7\alpha^3 + 6\alpha^4 + 5\alpha^5 + 4\alpha^6.$$

$$N_{K/\mathbb{Q}}(x_0 + x_1\alpha + \ldots + x_{n-1}\alpha^{n-1}) = m \quad \text{ in } x_0, x_1, \ldots, x_{n-1} \in \mathbb{Z}$$

where $K = \mathbb{Q}(\alpha)$ is an algebraic number field of degree n and m is a given integer such that $x_0, x_1, \ldots, x_{n-1}$ are consecutive terms in an arithmetic progression.

Norm form equations and arithmetic progressions

Bérczes and Pethő proved that this equation has only finitely many solutions if neither of the following two cases hold:

- α has minimal polynomial of the form

$$x^n - bx^{n-1} - \ldots - bx + (bn + b - 1)$$

  with $b \in \mathbb{Z}$.
- $\frac{n\alpha^n}{\alpha^n - 1} - \frac{\alpha}{\alpha - 1}$ is a real quadratic number.

Norm form equations and arithmetic progressions

Pethő's Problem 6

Does there exist infinitely many quartic algebraic integers α such that

$$\frac{4\alpha^4}{\alpha^4-1}-\frac{\alpha}{\alpha-1}$$

is a quadratic algebraic number?

The only example mentioned is $x^4+2x^3+5x^2+4x+2$ such that the corresponding element is a real quadratic number (that is a root of x^2-4x+2).

Norm form equations and arithmetic progressions

Maciej Ulas - Tengely (2017)

There are infinitely many quartic algebraic integers defined by $\alpha^4 + a\alpha^3 + b\alpha^2 + c\alpha + d = 0$ for which

$$\beta = \frac{4\alpha^4}{\alpha^4 - 1} - \frac{\alpha}{\alpha - 1}$$

is a quadratic algebraic number. Moreover, there are infinitely many quartic algebraic numbers α such that β is real quadratic.

Norm form equations and arithmetic progressions

- Let $t \in \mathbb{Z}$. The polynomials defined by

  $$f_1(x) = x^4 + 2x^3 + (2t^2+2)x^2 + (4t^2-4t+2)x + 6t^2 - 4t + 1$$

  are irreducible if $t \notin \{0,1\}$.
- Let $t \in \mathbb{Z}$. The polynomials defined by

  $$f_2(x) = x^4 + 2tx^3 + (t^2+2t+2)x^2 + (2t^2+2t)x + 3t^2 - 2t + 1$$

  are irreducible if $t \notin \{0,2\}$.

Norm form equations and arithmetic progressions

Let $f(x) = x^4 + ax^3 + bx^2 + cx + d$ with $a, b, c, d \in \mathbb{Z}$ and $g(x) = x^2 + px + q$ with $p, q \in \mathbb{Q}$. Assume that α is a root of $f(x)$ and $\beta = \frac{4\alpha^4}{\alpha^4-1} - \frac{\alpha}{\alpha-1}$ is a root of $g(x)$. From $g(\beta) = 0$ we get a degree 6 polynomial for which α is a root. Therefore it is divisible by $f(x)$. Computing the reminder we obtain a cubic polynomial which has to be zero.

Norm form equations and arithmetic progressions

$$
\begin{aligned}
e_1 : \quad & -3dpa^2 + 5dpa + 3dpb - 6dp - dqa^2 + 2dqa + dqb - 3dq - 9da^2 + 12da + 9db - 10d + q, \\
e_2 : \quad & 3dpa - 5dp + dqa - 2dq + 9da - 12d - 3pa^2c + 5pac + 3pbc - 6pc + p - qa^2c + 2qac + \\
& + qbc - 3qc + 2q - 9a^2c + 12ac + 9bc - 10c, \\
e_3 : \quad & -3dp - dq - 9d - 3pa^2b + 5pab + 3pac + 3pb^2 - 6pb - 5pc + 3p - qa^2b + 2qab + qac + \\
& + qb^2 - 3qb - 2qc + 3q - 9a^2b + 12ab + 9ac + 9b^2 - 10b - 12c + 1, \\
e_4 : \quad & -3pa^3 + 5pa^2 + 6pab - 6pa - 5pb - 3pc + 6p - qa^3 + 2qa^2 + 2qab - 3qa - 2qb - qc + 4q - \\
& - 9a^3 + 12a^2 + 18ab - 10a - 12b - 9c + 4.
\end{aligned}
$$

Norm form equations and arithmetic progressions

The Gröbner basis for $< e_1, e_2, e_3, e_4 >$ contains 19 polynomials, one of these factors as follows

$$
\left(\frac{1}{233}\right)\cdot(a-2b+c)\cdot
$$
$$
\cdot(233a^4-352a^3b+108a^3c+168a^3+368a^2b^2-264a^2bc-
$$
$$
-624a^2b+46a^2c^2-184a^2c-544a^2-160ab^3+128ab^2c+
$$
$$
352ab^2-16abc^2+64abc+128ab-4ac^3-8ac^2+768ac+
$$
$$
+640a+48b^4-64b^3c-256b^3+32b^2c^2+288b^2c+384b^2-
$$
$$
-8bc^3-144bc^2-512bc+c^4+24c^3+96c^2-640c-256).
$$

Special case $c = 2b - a$

Denote by $e_{1,c}, e_{2,c}, e_{3,c}, e_{4,c}$ the polynomials obtained by substituting $c = 2b - a$ into e_1, e_2, e_3 and e_4. Let us denote by G_c the Gröbner basis for $< e_{1,c}, e_{2,c}, e_{3,c}, e_{4,c} >$ and compute the ideal $I_{c,p,q} = G_c \cap \mathbb{Q}[a, b, d]$, i.e., we eliminate the variables p, q. The equation $(9b - 12a - 3d + 5)^2 - 4(3a - 2)^2 + 48d = 0$ defines the curve, say C, defined over $\mathbb{Q}(a)$ of genus 0 (in the plane (b, d)).

$$b = \frac{1}{36}(9a^2 + 36a - 16 - 8u - u^2), \quad d = \frac{1}{36}(9a^2 + 36a - 16 + 8u - u^2).$$

$$f(x) = \frac{1}{36}(6x + u + 3a - 2)\left(6x^3 + (3a - u + 2)x^2 + 2(3a - u - 1)x + 3(3a - u - 2)\right),$$

Difficult case, the quartic factor

$$\begin{aligned} F(a,b,c) \;=\; & 233a^4 - 352a^3b + 108a^3c + 168a^3 + 368a^2b^2 - 264a^2bc - \\ & -624a^2b + 46a^2c^2 - 184a^2c - 544a^2 - 160ab^3 + 128ab^2c + \\ & 352ab^2 - 16abc^2 + 64abc + 128ab - 4ac^3 - 8ac^2 + 768ac + \\ & +640a + 48b^4 - 64b^3c - 256b^3 + 32b^2c^2 + 288b^2c + 384b^2 - \\ & -8bc^3 - 144bc^2 - 512bc + c^4 + 24c^3 + 96c^2 - 640c - 256. \end{aligned}$$

$F(2,b,c)$ is a reducible polynomial given by

$$(12b^2-4bc-96b+c^2+12c+196)(4b^2-4bc-16b+c^2+4c+20).$$

$12b^2 - 4bc - 96b + c^2 + 12c + 196 = 0$

Consider the equation $12b^2 - 4bc - 96b + c^2 + 12c + 196 = 0$. It follows that $(c - 2b + 6)^2 + 2(2b - 9)^2 = 2$. The only integral solutions correspond with $b = 4$ or $b = 5$. If $b = 4$, then $c = 2$ and $d = 3$. We obtain the reducible polynomial $x^4 + 2x^2 + 4x^2 + 2x + 3 = (x^2 + 1)(x^2 + 2x + 3)$. If $b = 5$, then $c = 4$ and $d = 2$, the example from Pethő's paper.

$12b^2 - 4bc - 96b + c^2 + 12c + 196 = 0$

The set of rational solutions of $(c-2b+6)^2 + 2(2b-9)^2 = 2$ can be easily parametrized with

$$b = \frac{8t^2+5}{2t^2+1}, \quad c = \frac{4(t^2-t+1)}{2t^2+1}.$$

With $a = 2$ and b, c given above we easily compute the values

$$\begin{aligned} d &= \frac{2\left(3t^2-2t+1\right)}{2t^2+1}, \\ p &= \frac{4\left(2t^3-5t^2+t-1\right)}{4t^2+1}, \\ q &= -\frac{2(4t-1)\left(3t^2-2t+1\right)}{4t^2+1}. \end{aligned}$$

With p, q given above one can easily check that the discriminant of x^2+px+q is positive for all $t \in \mathbb{R}$ (and thus for all $t \in \mathbb{Q}$).

$4b^2 - 4bc - 16b + c^2 + 4c + 20 = 0$

We have $(2b-c)^2 + 20 = 4(4b-c)$. Let $u = 2b - c$ and $v = 4b - c$. We get that $v = \frac{u^2+20}{4}$ and $b = \frac{u^2-4u+20}{8}, c = \frac{u^2-8u+20}{4}$. Thus

$$\begin{aligned} b &= 2t^2 + 2, \\ c &= 4t^2 - 4t + 2, \end{aligned}$$

where $u = 4t + 2$. Let us denote by e_1', e_2', e_3' and e_4' the corresponding polynomials e_1, e_2, e_3 and e_4 after the substitution $a = 2, b = 2t^2 + 2, c = 4t^2 - 4t + 2$. We get that

$$G' \cap \mathbb{Q}[t][d] = < -t(1 - d - 4t + 6t^2)(t^3 + t^2 - t - 2), (d - 6t^2 + 4t - 1)(7 + d + 12t - 6t^2 - 8t^3) >$$

and thus $d = 6t^2 - 4t + 1$ or $t = 0$.

$4b^2 - 4bc - 16b + c^2 + 4c + 20 = 0$

If $t = 0$, then $d = 7$ and $f(x)$ is reducible $x^4 + 2x^3 + 2x^2 + 2x - 7 = (x-1)(x^3 + 3x^2 + 5x + 7)$, a contradiction. If $d = 6t^2 - 4t + 1$, then we have an infinite family of solutions of Pethő's question given by

$$
\begin{aligned}
a &= 2,\\
b &= 2t^2 + 2,\\
c &= 4t^2 - 4t + 2,\\
d &= 6t^2 - 4t + 1,\\
p &= -\frac{6\,t^2 - 6\,t + 1}{t^2 - t},\\
q &= \frac{18\,t^3 - 18\,t^2 + 7\,t - 1}{2\,(t^3 - t^2)}.
\end{aligned}
$$

Small solutions of $F(a, b, c) = 0$

$(-30, 197, 420, 706)$	$(-12, 26, 60, 121)$	$(6, 17, 24, 22)$
$(-28, 170, 364, 617)$	$(-10, 17, 40, 86)$	$(8, 26, 40, 41)$
$(-26, 145, 312, 534)$	$(-8, 10, 24, 57)$	$(10, 37, 60, 66)$
$(-24, 122, 264, 457)$	$(-6, 5, 12, 34)$	$(12, 50, 84, 97)$
$(-22, 101, 220, 386)$	$(-4, 2, 4, 17)$	$(14, 65, 112, 134)$
$(-20, 82, 180, 321)$	$(-2, 1, 0, 6)$	$(16, 82, 144, 177)$
$(-18, 65, 144, 262)$	$(0, 2, 0, 1)$	$(18, 101, 180, 226)$
$(-16, 50, 112, 209)$	$(2, 5, 4, 2)$	
$(-14, 37, 84, 162)$	$(4, 10, 12, 9)$	

Second family of solutions

All these solutions can be described by the formulas

$$\begin{aligned}
a &= 2t,\\
b &= t^2+2t+2,\\
c &= 2t^2+2t,\\
d &= 3t^2-2t+1,\\
p &= -\frac{2\left(3\,t^2-5\,t+4\right)}{t^2-2\,t+2},\\
q &= \frac{9\,t^3-12\,t^2+7\,t-2}{t^3-2\,t^2+2\,t}.
\end{aligned}$$

Second family of solutions

Let us note that the equation $F(a,b,c)=0$, defines (an affine) quartic surface, say V. The existence of the parametric solution presented above leads to the generic point (by taking $t=a/2$):

$$(a,b,c)=\left(a,\frac{a^2}{4}+a+2,\frac{a^2}{2}+a\right)$$

lying on V. This suggests to look on V as on a *quartic curve* defined over the rational function field $\mathbb{Q}(a)$. We call this curve $\mathcal{C}$. A quick computation in MAGMA reveals that the genus of $\mathcal{C}$ is 0. This implies that $\mathcal{C}$ is $\overline{\mathbb{Q}(a)}$-rational curve. Moreover, the existence of $\mathbb{Q}(a)$-rational point on $\mathcal{C}$ given by $P=\left(\frac{a^2}{4}+a+2,\frac{a^2}{2}+a\right)$ allows us to compute rational parametrization which is defined over $\mathbb{Q}(a)$.

Rational parametrization

$$b(t) = \frac{\sum_{i=0}^{6} bn_i(t)a^i}{\sum_{i=0}^{4} bd_i(t)a^i},$$

$$c(t) = \frac{\sum_{i=0}^{6} cn_i(t)a^i}{\sum_{i=0}^{4} cd_i(t)a^i},$$

$$d(t) = \frac{\sum_{i=0}^{6} dn_i(t)a^i}{\sum_{i=0}^{4} dd_i(t)a^i}.$$

i	$bn_i(t)$	$bd_i(t)$
0	$663552\,t^4 - 2211840\,t^3 + 2764800\,t^2 - 1536000\,t + 320000$	$331776\,t^4 - 1105920\,t^3 + 1382400\,t^2 - 768000\,t + 160000$
1	$-331776\,t^4 + 1050624\,t^3 - 1244160\,t^2 + 652800\,t - 128000$	$-331776\,t^4 + 1050624\,t^3 - 1244160\,t^2 + 652800\,t - 128000$
2	$-41472\,t^3 + 105984\,t^2 - 90240\,t + 25600$	$124416\,t^4 - 373248\,t^3 + 419328\,t^2 - 209280\,t + 39200$
3	$38016\,t^3 - 89280\,t^2 + 69696\,t - 18080$	$-20736\,t^4 + 58752\,t^3 - 62784\,t^2 + 30048\,t - 5440$
4	$12960\,t^4 - 47520\,t^3 + 62928\,t^2 - 36240\,t + 7748$	$1296\,t^4 - 3456\,t^3 + 3528\,t^2 - 1632\,t + 288$
5	$-3888\,t^4 + 11664\,t^3 - 13248\,t^2 + 6792\,t - 1332$	0
6	$324\,t^4 - 864\,t^3 + 900\,t^2 - 432\,t + 81$	0

Rational parametrization

i	$cn_i(t)$	$cd_i(t)$
0	0	$165888\,t^4 - 552960\,t^3 + 691200\,t^2 - 384000\,t + 80000$
1	$165888\,t^4 - 552960\,t^3 + 691200\,t^2 - 384000\,t + 80000$	$-165888\,t^4 + 525312\,t^3 - 622080\,t^2 + 326400\,t - 64000$
2	$-82944\,t^4 + 235008\,t^3 - 241920\,t^2 + 105600\,t - 16000$	$62208\,t^4 - 186624\,t^3 + 209664\,t^2 - 104640\,t + 19600$
3	$-20736\,t^4 + 86400\,t^3 - 126720\,t^2 + 79296\,t - 18080$	$-10368\,t^4 + 29376\,t^3 - 31392\,t^2 + 15024\,t - 2720$
4	$20736\,t^4 - 66528\,t^3 + 79920\,t^2 - 42744\,t + 8620$	$648\,t^4 - 1728\,t^3 + 1764\,t^2 - 816\,t + 144$
5	$-4536\,t^4 + 13176\,t^3 - 14580\,t^2 + 7308\,t - 1404$	0
6	$324\,t^4 - 864\,t^3 + 900\,t^2 - 432\,t + 81$	0

i	$cn_i(t)$	$cd_i(t)$
0	$331776\,t^4 - 1105920\,t^3 + 1382400\,t^2 - 768000\,t + 160000$	$331776\,t^4 - 1105920\,t^3 + 1382400\,t^2 - 768000\,t + 160000$
1	$-663552\,t^4 + 2211840\,t^3 - 2764800\,t^2 + 1536000\,t - 320000$	$-331776\,t^4 + 1050624\,t^3 - 1244160\,t^2 + 652800\,t - 128000$
2	$705024\,t^4 - 2350080\,t^3 + 2939904\,t^2 - 1635840\,t + 341600$	$124416\,t^4 - 373248\,t^3 + 419328\,t^2 - 209280\,t + 39200$
3	$-393984\,t^4 + 1271808\,t^3 - 1540224\,t^2 + 829536\,t - 167680$	$-20736\,t^4 + 58752\,t^3 - 62784\,t^2 + 30048\,t - 5440$
4	$115344\,t^4 - 353376\,t^3 + 407880\,t^2 - 210624\,t + 41148$	$1296\,t^4 - 3456\,t^3 + 3528\,t^2 - 1632\,t + 288$
5	$-16848\,t^4 + 48384\,t^3 - 52992\,t^2 + 26304\,t - 5004$	0
6	$972\,t^4 - 2592\,t^3 + 2700\,t^2 - 1296\,t + 243$	0

Second family of solutions

We were trying to use the obtained parametrization to find other integer points on the surface V but without success. If α is not an algebraic integer, then using the parametrizations we may obtain real quadratic algebraic numbers. Indeed, if α is a root of the polynomial $x^4+ax^3+bx^2+cx+d$ then write $\beta=\frac{4\alpha^4}{\alpha^4-1}-\frac{\alpha}{\alpha-1}$. As an example let us consider the case $a=1, t=1$. The formulas provide that α is a root of the polynomial

$$x^4+x^3+97/24x^2+3/4x+17/8$$

then β is a root of the following polynomial having two real roots

$$x^2-6/13x-51/5.$$

"Near misses" solutions of Pethő's problem

If α is solution of

$$x^4 + 2x^3 + \frac{14}{3}x^2 + 2x + 1$$

then β is root of the polynomial

$$x^2 + 3x - \frac{3}{4}.$$

Similarly, if α is a root of

$$x^4 + 2x^3 + \frac{13}{3}x^2 + 4x + 4$$

then β is a root of

$$x^2 + \frac{36}{5}x + 12.$$

2009. május 23.