Rational Functions and Arithmetic Progressions

Szabolcs Tengely

29th Journées Arithmétiques

9th July 2015

Research supported in part by the OTKA grants

K100339 and NK104208

Introduction

We are interested in $f \in k(x)$ that are decomposable as rational functions, i.e. for which

$$f(x) = g(h(x))$$

with $g, h \in k(x)$, $\deg g, \deg h \geq 2$ holds.
Such a decomposition is only unique up to a linear fractional transformation

$$\lambda = \frac{ax + b}{cx + d}$$

with $ad - bc = \pm 1$, since we may always replace $g(x)$ by $g(\lambda(x))$ and $h(x)$ by $\lambda^{-1}(h(x))$ without affecting the equation $f(x) = g(h(x))$.

Related results

1949: **Rényi** and **Erdős** conjectured independently: bound for the number of terms of $h(x)^2$ implies a bound for the terms of $h(x)$.
1987: **Schinzel** ingenious proof in the case $h(x)^d$.
Schinzel conjectured that if for fixed g the polynomial $g(h(x))$ has at most l non-constant terms, then the number of terms of h is bounded only in terms of l. A more general form of this conjecture was proved by **Zannier** in 2008.

1922: **Ritt** proved that if $f = p_1 \circ p_2 \circ \cdots p_s = q_1 \circ q_2 \circ \cdots q_r$, then $s = r$ and the sets of degrees of the polynomials are equal.

Extensions by **Beardon, Pakovich, Zieve** and many others.

1922: **Ritt** proved that if $f = p_1 \circ p_2 \circ \cdots p_s = q_1 \circ q_2 \circ \cdots q_r$, then $s = r$ and the sets of degrees of the polynomials are equal. Extensions by **Beardon, Pakovich, Zieve** and many others. It is not true that all complete decompositions of a rational function have the same length. **Gutierrez** and **Sevilla** provided an example with rational coefficients as follows

$$f = \frac{x^3(x+6)^3(x^2-6x+36)^3}{(x-3)^3(x^2+3x+9)^3},$$

$$f = g_1 \circ g_2 \circ g_3 = x^3 \circ \frac{x(x-12)}{x-3} \circ \frac{x(x+6)}{x-3},$$

$$f = h_1 \circ h_2 = \frac{x^3(x+24)}{x-3} \circ \frac{x(x^2-6x+36)}{x^2+3x+9}.$$

Several arithmetical applications, equations of type $f(x) = g(y)$:

- **Davenport, Lewis** and **Schinzel**
- **Fried**
- **Beukers, Shorey** and **Tijdeman**
- **Bilu** and **Tichy**
- **Győry**
- **Brindza** and **Pintér**

In this talk we are interested in rational functions

$$f = \frac{P}{Q}$$

with a **bounded number of zeros and poles** (i.e. the number of distinct roots of P, Q in a reduced expression of f is bounded).

We assume that the number of zeros and poles are fixed, whereas the actual values of the zeros and poles and their multiplicities are considered as variables.

Theorem by **Fuchs** and **Pethő**

Let n be a positive integer. Then there exists a positive integer $J \leq 2nn^{2n}$ and, for every $i \in \{1, \ldots, J\}$, an affine algebraic variety $\mathcal{V}_i$ defined over $\mathbb{Q}$ and with $\mathcal{V}_i \subset \mathbb{A}^{n+t_i}$ for some $2 \leq t_i \leq n$, such that:
(i) If $f, g, h \in k(x)$ with $f(x) = g(h(x))$ and with $\deg g, \deg h \geq 2$, g not of the shape $(\lambda(x))^m, m \in \mathbb{N}, \lambda \in \mathrm{PGL}_2(k)$, and f has at most n zeros and poles altogether, then there exists for some $i \in \{1, \ldots, J\}$ a point $P = (\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_{t_i}) \in \mathcal{V}_i(k)$,
a vector $(k_1, \ldots, k_{t_i}) \in \mathbb{Z}^{t_i}$ with $k_1 + k_2 + \ldots + k_{t_i} = 0$ or not depending on $\mathcal{V}_i$, a partition of $\{1, \ldots, n\}$ in $t_i + 1$ disjoint sets $S_\infty, S_{\beta_1}, \ldots, S_{\beta_{t_i}}$ with $S_\infty = \emptyset$ if $k_1 + k_2 + \cdots + k_{t_i} = 0$,
and a vector $(l_1, \ldots, l_n) \in \{0, 1, \ldots, n-1\}^n$, also both depending only on $\mathcal{V}_i$, such that

$$f(x)=\prod_{j=1}^{t_i}(w_j/w_\infty)^{k_j}, \qquad g(x)=\prod_{j=1}^{t_i}(x-\beta_j)^{k_j},$$

and

$$h(x)=\begin{cases}\beta_j+\frac{w_j}{w_\infty}\ (j=1,\ldots,t_i), & \text{if } k_1+k_2+\cdots+k_{t_i}\neq 0,\\ \frac{\beta_{j_1}w_{j_2}-\beta_{j_2}w_{j_1}}{w_{j_2}-w_{j_1}}\ (1\le j_1<j_2\le t_i), & \text{otherwise},\end{cases}$$

where

$$w_j = \prod_{m\in S_{\beta_j}}(x-\alpha_m)^{l_m}, \quad j=1,\ldots,t_i,$$
$$w_\infty = \prod_{m\in S_\infty}(x-\alpha_m)^{l_m}.$$

Moreover, we have $\deg h \le (n-1)/(t_i-1)\le n-1$.

(ii) Conversely for given data $P \in \mathcal{V}_i(k), (k_1, \dots, k_{t_i}), (l_1, \dots, l_n)$, $S_\infty, S_{\beta_1} \dots, S_{\beta_{t_i}}$, as described in (i) one defines by the same equations rational functions f, g, h with f having at most n zeros and poles altogether for which $f(x) = g(h(x))$ holds.

(iii) The integer J and equations defining the varieties $\mathcal{V}_i$ are effectively computable only in terms of n.

Tools from the theory of valuation

The **Mason-Stothers (1984) theorem** says: Let $f, g \in k(x)$, *not both constant and let* S *be any set of valuations of* $k(x)$ *containing all the zeros and poles in* $\mathbb{P}^1(k)$ *of* f *and* g*. Then we have* $\max\{\deg f, \deg g\} \leq |S| - 2$. Best possible.

More generally **Zannier** (1995) proved: Let S is any set of valuations of $k(x)$ containing all the zeros and poles in $\mathbb{P}^1(k)$ of $g_1, \ldots, g_m$. If $g_1, \ldots, g_m \in k(x)$ *span a* k*-vector space of dimension* $\mu < m$ *and any* μ *of the* g_i *are linearly independent over* k*, then*

$$-\sum_{v \in \mathcal{M}} \min\{v(g_1), \ldots, v(g_m)\} \leq \frac{1}{m-\mu}\binom{\mu}{2}(|S| - 2).$$

Since k is algebraically closed we can write

$$f(x) = \prod_{i=1}^{n} (x - \alpha_i)^{f_i}$$

with pairwise distinct $\alpha_i \in k$ and $f_i \in \mathbb{Z}$ for $i = 1, \ldots, n$.
Similarly we get

$$g(x) = \prod_{j=1}^{t} (x - \beta_j)^{k_j}$$

with pairwise distinct $\beta_j \in k$ and $k_j \in \mathbb{Z}$ for $j = 1, \ldots, t$ and $t \in \mathbb{N}$.
Thus we have

$$\prod_{i=1}^{n} (x - \alpha_i)^{f_i} = f(x) = g(h(x)) = \prod_{j=1}^{t} (h(x) - \beta_j)^{k_j}.$$

We now distinguish two cases depending on $k_1 + k_2 + \cdots + k_t \neq 0$ or not; observe that this condition is equivalent to $v_\infty(g) \neq 0$ or not.
We shall write $h(x) = p(x)/q(x)$ with $p, q \in k[x]$, p, q coprime.

The case $k_1 + k_2 + \cdots + k_t \neq 0$

There is a subset S_∞ of the set $\{1, \ldots, n\}$ such that the α_m for $m \in S_\infty$ are precisely the poles in $\mathbb{A}^1(k)$ of h, i.e.

$$q(x) = \prod_{m \in S_\infty} (x - \alpha_m)^{l_m}, \; l_m \in \mathbb{N}.$$

Furthermore h and $h(x) - \beta_j$ have the same number of poles counted by multiplicity, which means that their degrees are equal.

There is a partition of the set $\{1, \ldots, n\} \setminus S_\infty$ in t disjoint subsets $S_{\beta_1}, \ldots, S_{\beta_t}$ such that

$$h(x) = \beta_j + \frac{1}{q(x)} \prod_{m \in S_{\beta_j}} (x - \alpha_m)^{l_m},$$

where $l_m \in \mathbb{N}$ satisfies $l_m k_j = f_m$ for $m \in S_{\beta_j}, j = 1, \ldots, t$.

Since we assume that g is not of the shape $(\lambda(x))^m$ it follows that $t \geq 2$. Let $1 \leq i < j \leq t$ be given. We have at least two different representations of h and thus we get

$$\beta_i + \frac{1}{q(x)} \prod_{r \in S_{\beta_i}} (x - \alpha_r)^{l_r} = \beta_j + \frac{1}{q(x)} \prod_{s \in S_{\beta_j}} (x - \alpha_s)^{l_s}$$

or equivalently $\beta(u_i - u_j) = 1$, where $\beta = 1/(\beta_j - \beta_i)$ and

$$u_i = \frac{1}{q(x)} \prod_{r \in S_{\beta_i}} (x - \alpha_r)^{l_r} = \frac{w_i}{w_\infty}.$$

The case $k_1 + k_2 + \cdots + k_t = 0$

Here we have

$$\prod_{i=1}^{n}(x-\alpha_i)^{f_i} = \prod_{j=1}^{t}\left(\frac{p(x)}{q(x)}-\beta_j\right)^{k_j} = \prod_{j=1}^{t}(p(x)-\beta_j q(x))^{k_j}.$$

There is a partition of the set $\{1,\ldots,n\}$ in t disjoint subsets $S_{\beta_1},\ldots,S_{\beta_t}$ such that

$$(p(x)-\beta_j q(x))^{k_j} = \prod_{m\in S_{\beta_j}}(x-\alpha_m)^{f_m}.$$

Thus k_j divides f_m for all $m \in S_{\beta_j}, j = 1,\ldots,t$. On putting $l_m = f_m/k_j$ for $m \in S_{\beta_j}$ we obtain

$$p(x)-\beta_j q(x) = \prod_{m\in S_{\beta_j}}(x-\alpha_m)^{l_m}, j = 1,\ldots,t.$$

Let us choose $1 \leq j_1 < j_2 < j_3 \leq t$. From the corresponding three equations the so called **Siegel identity** $v_{j_1,j_2,j_3} + v_{j_3,j_1,j_2} + v_{j_2,j_3,j_1} = 0$ follows, where

$$v_{j_1,j_2,j_3} = (\beta_{j_1} - \beta_{j_2}) \prod_{m \in S_{\beta_{j_3}}} (x - \alpha_m)^{l_m}.$$

The quantities v_{j_1,j_2,j_3} are non-constant rational functions and they are S-units. Observe that by taking $j_1 = 1, j_2 = i, j_3 = j$ with $1 \leq i < j \leq t$ the Siegel identity can be rewritten as

$$\frac{\beta_j - \beta_1}{\beta_j - \beta_i} \frac{w_i}{w_1} + \frac{\beta_1 - \beta_i}{\beta_j - \beta_i} \frac{w_j}{w_1} = 1.$$

An algorithm to compute solutions

Pethő and **Tengely** provided and algorithm implemented in MAGMA:
1) Let $S_\infty, S_{\beta_1}, \dots, S_{\beta_t}$ be a partition of $\{1, 2, \dots, n\}$.
2) For the partition and a vector $(l_1, \dots, l_n) \in \{0, 1, \dots, n-1\}^n$ compute the corresponding variety $V = \{v_1, \dots, v_r\}$, where v_i is a polynomial in $\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_t$. Here we used Groebner basis technique.
3) To remove contradictory systems we compute
$\Phi = \prod_{i \neq j}(\alpha_i - \alpha_j) \prod_{i \neq j}(\beta_i - \beta_j)$.
4) For all v_i compute

$$u_{i_1} = \frac{v_i}{\gcd(v_i, \Phi)},$$

and

$$u_{i_k} = \frac{u_{i_{k-1}}}{\gcd(u_{i_{k-1}}, \Phi)},$$

until $\gcd(u_{i_{k-1}}, \Phi) = 1$.

We note that Ayad and Fleischmann implemented a MAGMA code to find decompositions of a given rational function, as an example they considered the rational function

$$f = \frac{x^4 - 8x}{x^3 + 1}$$

and they obtained that $f(x) = g(h(x))$, where

$$g = \frac{x^2 + 4x}{x + 1} \quad \text{and} \quad h = \frac{x^2 - 2x}{x + 1}.$$

Using our MAGMA procedure

```
CFunc(3,7,1:PSet:=[[{1},{2,3},{4,5},{6,7}]],exptup:=[[1,1,1,1,1,1,1]]);
```

we get the system of equations

$$
\begin{aligned}
\alpha_1\beta_1 - \alpha_1\beta_3 + \alpha_3^2 - \alpha_3\alpha_6 - \alpha_3\alpha_7 - \alpha_3\beta_1 + \alpha_3\beta_3 + \alpha_6\alpha_7 &= 0,\\
\alpha_1\beta_2 - \alpha_1\beta_3 + \alpha_5^2 - \alpha_5\alpha_6 - \alpha_5\alpha_7 - \alpha_5\beta_2 + \alpha_5\beta_3 + \alpha_6\alpha_7 &= 0,\\
\alpha_2 + \alpha_3 - \alpha_6 - \alpha_7 - \beta_1 + \beta_3 &= 0,\\
\alpha_3^2\beta_2 - \alpha_3^2\beta_3 - \alpha_3\alpha_6\beta_2 + \alpha_3\alpha_6\beta_3 - \alpha_3\alpha_7\beta_2 + \alpha_3\alpha_7\beta_3 - &\\
\alpha_3\beta_1\beta_2 + \alpha_3\beta_1\beta_3 + \alpha_3\beta_2\beta_3 - \alpha_3\beta_3^2 - \alpha_5^2\beta_1 + \alpha_5^2\beta_3 + &\\
\alpha_5\alpha_6\beta_1 - \alpha_5\alpha_6\beta_3 + \alpha_5\alpha_7\beta_1 - \alpha_5\alpha_7\beta_3 + \alpha_5\beta_1\beta_2 - \alpha_5\beta_1\beta_3 - &\\
\alpha_5\beta_2\beta_3 + \alpha_5\beta_3^2 - \alpha_6\alpha_7\beta_1 + \alpha_6\alpha_7\beta_2 &= 0,\\
\alpha_4 + \alpha_5 - \alpha_6 - \alpha_7 - \beta_2 + \beta_3 &= 0.
\end{aligned}
$$

Using our MAGMA procedure

```
CFunc(3,7,1:PSet:=[[{1},{2,3},{4,5},{6,7}]],exptup:=[[1,1,1,1,1,1,1]]);
```

we get the system of equations

$$
\begin{aligned}
\alpha_1\beta_1 - \alpha_1\beta_3 + \alpha_3^2 - \alpha_3\alpha_6 - \alpha_3\alpha_7 - \alpha_3\beta_1 + \alpha_3\beta_3 + \alpha_6\alpha_7 &= 0,\\
\alpha_1\beta_2 - \alpha_1\beta_3 + \alpha_5^2 - \alpha_5\alpha_6 - \alpha_5\alpha_7 - \alpha_5\beta_2 + \alpha_5\beta_3 + \alpha_6\alpha_7 &= 0,\\
\alpha_2 + \alpha_3 - \alpha_6 - \alpha_7 - \beta_1 + \beta_3 &= 0,\\
\alpha_3^2\beta_2 - \alpha_3^2\beta_3 - \alpha_3\alpha_6\beta_2 + \alpha_3\alpha_6\beta_3 - \alpha_3\alpha_7\beta_2 + \alpha_3\alpha_7\beta_3 - &\\
\alpha_3\beta_1\beta_2 + \alpha_3\beta_1\beta_3 + \alpha_3\beta_2\beta_3 - \alpha_3\beta_3^2 - \alpha_5^2\beta_1 + \alpha_5^2\beta_3 + &\\
\alpha_5\alpha_6\beta_1 - \alpha_5\alpha_6\beta_3 + \alpha_5\alpha_7\beta_1 - \alpha_5\alpha_7\beta_3 + \alpha_5\beta_1\beta_2 - \alpha_5\beta_1\beta_3 - &\\
\alpha_5\beta_2\beta_3 + \alpha_5\beta_3^2 - \alpha_6\alpha_7\beta_1 + \alpha_6\alpha_7\beta_2 &= 0,\\
\alpha_4 + \alpha_5 - \alpha_6 - \alpha_7 - \beta_2 + \beta_3 &= 0.
\end{aligned}
$$

We note that the above system has a solution

$$
\begin{aligned}
&(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \beta_1, \beta_2, \beta_3) =\\
&(-1, 0, 2, -1 - \sqrt{-3}, -1 + \sqrt{-3}, \frac{1 - \sqrt{-3}}{2}, \frac{1 + \sqrt{-3}}{2}, 0, -4, -1).
\end{aligned}
$$

It corresponds to the example given by Ayad and Fleischmann, that is

$$
f = \frac{x^4 - 8x}{x^3 + 1}, \quad g = \frac{x^2 + 4x}{x + 1}, \quad h = \frac{x^2 - 2x}{x + 1}.
$$

Let k be an algebraically closed field of characteristic zero. **Pethő** and **Tengely** provided two families of decomposable rational functions having 3 zeros and poles altogether.

(a) $\frac{(x-\alpha_1)^{k_1}(x+1/4-\alpha_1)^{2k_2}}{(x-1/4-\alpha_1)^{2k_1+2k_2}}$ for some $\alpha_1 \in k$ and $k_1, k_2 \in \mathbb{Z}$, $k_1 + k_2 \neq 0$,

(b) $\frac{(x-\alpha_1)^{2k_1}(x+\alpha_1-2\alpha_2)^{2k_2}}{(x-\alpha_2)^{2k_1+2k_2}}$ for some $\alpha_1, \alpha_2 \in k$ and $k_1, k_2 \in \mathbb{Z}$, $k_1 + k_2 \neq 0$.

Let k be an algebraically closed field of characteristic zero. **Pethő** and **Tengely** provided two families of decomposable rational functions having 3 zeros and poles altogether.

(a) $\frac{(x-\alpha_1)^{k_1}(x+1/4-\alpha_1)^{2k_2}}{(x-1/4-\alpha_1)^{2k_1+2k_2}}$ for some $\alpha_1 \in k$ and $k_1, k_2 \in \mathbb{Z}$, $k_1 + k_2 \neq 0$,

(b) $\frac{(x-\alpha_1)^{2k_1}(x+\alpha_1-2\alpha_2)^{2k_2}}{(x-\alpha_2)^{2k_1+2k_2}}$ for some $\alpha_1, \alpha_2 \in k$ and $k_1, k_2 \in \mathbb{Z}$, $k_1 + k_2 \neq 0$.

We note that in both cases the zeros and poles form an arithmetic progression:

$$\alpha_1 - \frac{1}{4}, \alpha_1, \alpha_1 + \frac{1}{4}, \quad \text{difference} = \frac{1}{4}$$

and

$$\alpha_1, \alpha_2, 2\alpha_2 - \alpha_1, \quad \text{difference} = \alpha_2 - \alpha_1.$$

Problem: determine decomposable rational functions having zeros and poles forming an arithmetic progression.
If $S_\infty \neq \emptyset$, then we take S_{β_i} and S_{β_j} two partitions having minimal cardinality. The zeros and poles satisfy $\alpha_i = \alpha_0 + k_i d$.
We have that

$$\beta_i - \beta_j = \frac{\prod_{s \in S_{\beta_j}} (\alpha_r - \alpha_s)^{l_s}}{\prod_{m \in S_\infty} (\alpha_r - \alpha_m)^{l_m}}$$

$$\beta_i - \beta_j = -\frac{\prod_{r \in S_{\beta_i}} (\alpha_s - \alpha_r)^{l_r}}{\prod_{m \in S_\infty} (\alpha_s - \alpha_m)^{l_m}}.$$

Hence

$$\frac{u_1 d^{v_1}}{u_2 d^{v_2}} = -\frac{u_3 d^{v_3}}{u_4 d^{v_2}} \Rightarrow -\frac{u_1 u_4}{u_2 u_3} = d^{v_3 - v_1}.$$

If $S_\infty = \emptyset$, then we take $S_{\beta_{j_1}}, S_{\beta_{j_2}}$ and $S_{\beta_{j_3}}$ three partitions having minimal cardinality. **Siegel identity** yields

$$\begin{aligned} v_{j_1,j_2,j_3}(\alpha_{j_1}) + v_{j_3,j_1,j_2}(\alpha_{j_1}) &= 0 \\ v_{j_1,j_2,j_3}(\alpha_{j_2}) + v_{j_2,j_3,j_1}(\alpha_{j_2}) &= 0 \\ v_{j_3,j_1,j_2}(\alpha_{j_3}) + v_{j_2,j_3,j_1}(\alpha_{j_3}) &= 0. \end{aligned}$$

After eliminating $\beta_{j_1}, \beta_{j_2}, \beta_{j_3}$ one obtains that

$$\gamma = d^\delta.$$

In both cases we get a finite list of possible values of d and a finite list of special tuples $(k_1, \ldots, k_n, l_1, \ldots, l_n)$ for which $v_3 - v_1 = 0$ and $-\frac{u_1 u_4}{u_2 u_3} = 1$ or $\delta = 0$ and $\gamma = 1$.

Example

Let $n = 5$ and $|S_\infty| = 1, |S_{\beta_1}| = |S_{\beta_2}| = 2$. We have the following system of equations

$$\begin{aligned}
x = \alpha_1 : \quad \beta_1 - \beta_2 &= \frac{(k_1 - k_3)^{l_3}(k_1 - k_4)^{l_4} d^{l_3+l_4}}{(k_1 - k_5)^{l_5} d^{l_5}} \\
x = \alpha_2 : \quad \beta_1 - \beta_2 &= \frac{(k_2 - k_3)^{l_3}(k_2 - k_4)^{l_4} d^{l_3+l_4}}{(k_2 - k_5)^{l_5} d^{l_5}} \\
x = \alpha_3 : \quad \beta_1 - \beta_2 &= -\frac{(k_3 - k_1)^{l_1}(k_3 - k_2)^{l_2} d^{l_1+l_2}}{(k_3 - k_5)^{l_5} d^{l_5}} \\
x = \alpha_4 : \quad \beta_1 - \beta_2 &= -\frac{(k_4 - k_1)^{l_1}(k_4 - k_2)^{l_2} d^{l_1+l_2}}{(k_4 - k_5)^{l_5} d^{l_5}}.
\end{aligned}$$

If $l_1 + l_2 - l_3 - l_4 \neq 0$, then d is an element of a finite set having 100 elements.

Example

In case $(k_1, k_2, k_3, k_4, k_5) = (0, 4, 1, 3, 2)$ and $(l_1, l_2, l_3, l_4, l_5) = (1, 1, 2, 2, 2)$ we have that $d = \frac{2\sqrt{3}}{3}$ and

$$
\begin{aligned}
g(x) &= (x - \beta_1)(x - \beta_1 + 3) \\
h(x) &= \beta_1 + \frac{(x - \alpha_0)(x - \alpha_0 - 4d)}{(x - \alpha_0 - 2d)^2} \\
f(x) &= \frac{4(x - \alpha_0)(x - \alpha_0 - d)(x - \alpha_0 - 3d)(x - \alpha_0 - 4d)}{(x - \alpha_0 - 2d)^4}.
\end{aligned}
$$

Example

In case $(k_1, k_2, k_3, k_4, k_5) = (1, 4, 3, 0, 2)$ and $(l_1, l_2, l_3, l_4, l_5) = (1, 1, 1, 1, 1)$ we have that $l_1 + l_2 - l_3 - l_4 = 0$ and

$$\begin{aligned} g(x) &= (x - \beta_1)(x - \beta_1 + 2d) \\ h(x) &= \beta_1 + \frac{(x - \alpha_0 - d)(x - \alpha_0 - 4d)}{(x - \alpha_0 - 2d)} \\ f(x) &= \frac{(x - \alpha_0)(x - \alpha_0 - d)(x - \alpha_0 - 3d)(x - \alpha_0 - 4d)}{(x - \alpha_0 - 2d)}. \end{aligned}$$

My favorite sequence

Number Theory and Its
Applications
October 4-8, 2010.
Debrecen

29th Journées Arithmétiques
July 6-10, 2015.
Debrecen

My favorite sequence

Number Theory and Its Applications
October 4-8, 2010.
Debrecen

Hanna Tengely: October 2, 2010.

29th Journées Arithmétiques
July 6-10, 2015.
Debrecen

Sára Tengely: July 4, 2015.