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Diophantine Equations Related to Linear Recurrence Sequences

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Dissertation for the degree of Doctor of Philosophy (PhD)

UNIVERSITY OF DEBRECEN

Doctoral Council of Natural Sciences

Doctoral School of Mathematical and Computational Sciences

Debrecen, 2021

Hereby I declare that I prepared this dissertation within the Doctoral Council of Natural Sciences, Doctoral School of Mathematical and Computational Sciences, University of Debrecen in order to obtain a PhD Degree in Natural Sciences, Mathematics at University of Debrecen.

The results published in the dissertation have not been reported in any other PhD dissertations.

Debrecen, March 4, 2021.

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Hereby I confirm that the candidate Hayder Raheem Hashim conducted his studies with my supervision within the Diophantine and Constructive Number Theory Doctoral Program of the Doctoral School of Mathematical and Computational Sciences, University of Debrecen between 2017 and 2021. The independent studies and research work of the candidate are significantly contributed to the results published in the dissertation.

I also declare that the results published in the dissertation have not been reported in any other dissertations.

I support the acceptance of the dissertation.

Debrecen, March 4, 2021.

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Diophantine equations related to linear recurrence sequences

Dissertation submitted in partial fulfilment of the requirements for the doctoral (PhD) degree in Mathematics.

Written by: Hayder Raheem Hashim certified mathematician

Prepared in the framework of the Diophantine and Constructive Number Theory program of the Doctoral School of Mathematical and Computational Sciences of the University of Debrecen.

Dissertation advisor: Dr. Szabolcs Tengely

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	Dr.	....

The date of the dissertation defence: .... 2021.

" Research is to see what everybody else has seen, and to think what nobody else has thought."

Albert Szent-Györgyi

Acknowledgments

Words are certainly not enough to express my gratitude and appreciation to those who have helped, supported and encouraged me throughout the years of my PhD studies at University of Debrecen / Hungary.

From the bottom of my heart, I would like to say a big and special thank you to my patient and supportive supervisor Dr. Szabolcs Tengely for his aspiring guidance, consistent encouragement and continuous support throughout my PhD studies. Without his generous help and valuable advices, this dissertation could not have come about. It was a great pleasure for me to work with him and be under his supervision. I am really indebted to him more than he knows.

I would also like to thank the head of Doctoral School of Mathematical and Computational Sciences, Prof. Dr. Páles Zsolt for his usual help, guidance and continuous support during my years of study.

I am also extremely thankful to the director of Institute of Mathematics, Prof. Dr. Hajdu Lajos, to the head of Department of Algebra and Number Theory, Prof. Dr. Bérczes Attila, and to the coordinator of Doctoral School of Mathematical and Computational Sciences, Prof. Dr. Magdolna Várterész for their valuable guidance, timely suggestions and help since the first day of my PhD program.

I would also like to express my thankfulness towards everybody at the Department of Algebra and Number Theory as well as in the other departments, especially Prof. Dr. Gaál István, Dr. Györkös-Varga Nóra, Komjáthi Barnáné, Kovácsné Kiss Adrienn and Homonnainé Bajkai Judit for helping me at their best at various times and for making the environment so congenial to work at every step in one way or the other. People here are genuinely nice and helpful, and I am glad to have interacted with many.

My sincere thanks go to all of my colleagues at the Institute of Mathematics including Muwafaq Salih, Amr Zakaria and Remete László who have always helped and supported me in all aspects during my university years in Debrecen. I consider myself fortunate to have such helpful and understanding friends.

I gratefully acknowledge my words of gratitude to the Tempus Public Foundation "Stipendium Hungaricum Scholarship Programme" and to the Iraqi Ministry of Higher Education and Scientific Research / University of Kufa for providing me this opportunity and supporting me during my years of study.

No words are enough to thank my lovely parents, brothers and sister who completely dedicated their lives into making mine better. I am deeply grateful to each and every one of them for their unending support, encouragement and prayers.

I indeed owe gigantic thanks to a very special person in my life, my lovely wife Zahraa for her continued and unfailing love, usual support, encouragement, patience and understanding throughout the years. I truly appreciate my first handsome son, Mujtaba who was born on September 18, 2017 that was my first day as a PhD student at University of Debrecen. Since then, he has made my life full of joy and happiness. I must say he is the best part of my life. Words would never say how grateful I am to both of you, and thanks for making my dreams come true.

Last but not least, I would like to thank all of those people unnamed for their moral support and those who helped me directly and indirectly to complete my PhD work.

Contents

CHAPTER 1

Introduction and preliminaries

1.1. History

The theory of Diophantine equations is a classical subject that involves solving an equation of the form $f(x_1, x_2, \ldots, x_n) = 0$, where f is an n-variable function and $x_1, x_2, \ldots, x_n$ are required to be integers or rational numbers. Many methods have been devoted to the study of Diophantine equations theoretically, an observation which may come with no surprise given that the study of such equations goes back thousands of years. Concerning a Diophantine equation, there are some questions one should arise:

- Is the equation solvable ?
- If it is solvable, is the solution set to this equation finite or infinite ?
- If it is solvable, is it possible to give an effective procedure to find all the solutions ?

Indeed, there are many questions that one can ask regarding the solutions of Diophantine equations. In this dissertation, we will study the solutions of some types of Diophantine equations concerning linear recurrence sequences. Furthermore, we give some strategies to completely determine such solutions. Before presenting our main results, we introduce a historical survey related to Diophantine equations with some relevant problems, concepts and notations, which will be used throughout the dissertation. The name of these equations is in honor of the mathematician Diophantus who did his work and lived in Alexandria around 300 AD. He was well known by his ancient text Diophantus' Arithmetica which was comprised of thirteen books, but only six Greek manuscripts have survived into the modern era, and they have been edited and translated several times. In fact, one of the ancient Diophantine equations presented by the equation

$$x^n + y^n = z^n, \tag{1}$$

where all the variables including n are rational integers. If $n = 1$, equation (1) is trivial since it stands as the sum of any two integers is clearly an integer. So we have $x + y = z$. On the other hand, if $n = 2$, equation (1) presents Pythagorean equation: $x^2 + y^2 = z^2$, which was solved by the Greek mathematician Pythagoras who lived around 500 BC. However, around 1500 BC the Babylonians were aware of the solution $(4961, 6480, 8161)$. Pythagoras stated and proved his theorem in the following form:

THEOREM 1.1. *The square of the hypotenuse is equal to the sum of the squares of the other two sides.*

It is easy to show that there are infinitely many positive integer solutions of the Pythagorean equation. Indeed, if (x_0, y_0, z_0) is an arbitrary solution, then there are infinitely many solutions for this equation of the form (cx_0, cy_0, cz_0) for any nonnegative integer c.

In case of $n > 2$, equation (1) has no integer solutions such that $xyz \neq 0$ and this presents what is called Fermat's last theorem (sometimes called Fermat's conjecture), that was written by Pierre de Fermat $(1601-1665)$ by the following note in the margin of his edition of Diophantus' Arithmetica:

" CUBUM AUTEM IN DUOS CUBOS, AUT QUADRATO QUADRATUM IN DUOSQUADRATO QUADRATOS, ET GENERALITER NULLAM IN INFINITUM ULTRA QUA-DRATUM POTESTATEM IN DUOS EJUSDEM NOMINIS FAS EST DIVIDERE; CUJUSREI DEMONSTRATIONEM MIRABILE SANE DETEXI. HANC MARGINIS EXIGUITASNON CAPERET."

He claimed to have a proof of this conjecture, but it was never found. In fact, this statement was published by his son after his death. Moreover, Fermat proved that the equation $x^4 + y^4 = z^2$ has no nontrivial solutions using his method of infinite descent, which was not so well known. This method was found in a letter entitled "Relation des nouvelles decouvertes en la science des nombres" that he wrote to Pierre de Carcavi in 1659 (see e.g. [**69**]), telling him that he has discovered a new method of demonstration, which can be applied to many problems in the theory of number.

In case of $n = 4$, equation (1) can be written as $x^4 + y^4 = (z^2)^2$, so the infinite descent proves Fermat's conjecture for $n = 4$, see e.g. [**16**], [**74**], [**95**] and the references given there.

One of the most well known results on Fermat's last theorem was given around 1820 by Sophie Germain where n is an odd prime number, and the result stated in the following theorem, that was named after her (see e.g. [**140**]).

THEOREM 1.2. *If l is a prime number greater than 2 such that $q = 2l + 1$ is also prime, then the equation*

$$x^l + y^l = z^l \tag{2}$$

has no solutions in nonzero integers x, y, z, which are not divisible by l.

This theorem was considered as the first general proposition on Fermat's last theorem. In fact, Germain's theorem was called the first case of Fermat's last theorem, and the case where equation (2) has no solutions in nonzero integers x, y, z in which at least one of them is divisible by l is considered as the second case. These cases were studied by many mathematicians who realized that these two cases have roughly the same difficulties in their proofs, although in the first case simpler techniques were used. By using similar techniques used by Germain, Terjanian [**245**] proved the following theorem, that shows the Germain's theorem is possible if l is replaced by an even exponent.

THEOREM 1.3. *If the equation*

$$x^{2l} + y^{2l} = z^{2l}$$

has a nonzero solution in integers x, y, z, then either x or y is divisible by $2l$, where $l \geq 3$ is a prime number.

In addition, elegant proofs were given by Kummer for these two cases of Fermat's last theorem. It was the first time of using some important concepts of algebraic number theory such as cyclotomic fields in the proof of this theorem. He in fact showed that how the subject of the number of divisor classes of cyclotomic fields leads to the proof of Fermat's last theorem at these cases. His approach mainly depends on the factorization of equation (2) into the form

$$\prod_{k=0}^{l-1}(x + \zeta_l^k y) = z^l$$

over the ring $\mathbb{Z}[\zeta_l]$ generated by the l^{th} roots of unity. For more details about Kummer's approach in the proofs of the cases of Fermat's last theorem, one can see e.g. [**32**], [**65**], [**66**], [**255**] and the references given there.

The first proof of Fermat's conjecture was discovered in 1994 and published in 1995 by Wiles [**257**] while working on a more general problem in geometry in a joint work with Taylor in [**239**]. His proof was completely based on elliptic curves, which is an important field of Diophantine number theory, and the proof was a consequence of proving that every semistable elliptic curve over $\mathbb{Q}$ is modular. In the following we summarize Wiles' approach for proving Fermat's last theorem. We start by recalling some important concepts used in the proof of this theorem. Indeed, some of these concepts may appear later in connection with our main results.

First of all, we define elliptic curves and the modularity and semistability of elliptic curves. In general, curves are very important for a lot of reasons. For instance, in the curves of genus zero, rational curves, everything is algorithmic and they can be parameterized by rational functions, so they can be understood very well. Moreover, the curves of genus $g \geq 2$ are not easy to handle, however curves of genus $g = 1$ that can be parameterized by elliptic functions called the elliptic curves. Elliptic curves have a very rich structure since they have a natural group law, and they appeared in the study of Diophantine equations and their history dates back to ancient Greece and beyond. They have occurred in these equations with two different approaches, one is algebraic number theory, and the other is the analysis of algebraic varieties. The first one uses properties of the rings and fields for which the solutions lie in, whereas the analysis of algebraic varieties studies geometric objects. For later use we also recall the following. The points on a curve where both partial derivatives vanish are called singular points on the curve. Hence, a curve with no singular points is called a non-singular curve. It is known that a cubic curve is a projective curve of degree 3, and the general homogeneous cubic polynomial

$$F(X:Y:Z) = c_0X^3 + c_1X^2Y + c_2XY^2 + c_3Y^3 + c_4X^2Z + c_5XYZ + \\ + c_6Y^2Z + c_7XZ^2 + c_8YZ^2 + c_9Z^3,$$

with constants $c_0, ..., c_9$ in some field $\mathbb{K}$, for which $F = 0$ defines an elliptic curve $\mathbb{E}$. Moreover, the most general definition of a curve $\mathbb{E}$ over a field $\mathbb{K}$ given by the nonsingular generalized Weierstrass equation in its affine form

$$y^2 + \alpha_1xy + \alpha_3y = x^3 + \alpha_2x^2 + \alpha_4x + \alpha_6, \tag{3}$$

where $\alpha_1, \ldots, \alpha_6 \in \mathbb{K}$ with nonzero discriminant Δ, which is defined as follows

$$\left\{\begin{array}{lc} \Delta = & -\beta_2^2\beta_8 - 8\beta_4^3 - 27\beta_6^2 + 9\beta_2\beta_4\beta_6, \\ \beta_2 = & \alpha_1^2 + 4\alpha_2, \\ \beta_4 = & 2\alpha_4 + \alpha_1\alpha_3, \\ \beta_6 = & \alpha_3^2 + 4\alpha_6, \\ \beta_8 = & \alpha_1^2\alpha_6 + 4\alpha_2\alpha_6 - \alpha_1\alpha_3\alpha_4 + \alpha_2\alpha_3^2 - \alpha_4^2. \end{array}\right\}$$

Indeed, this form is useful when working with fields of characteristic 2 and characteristic 3. If we work with fields with characteristic different of 2, then dividing equation (3) by 2 and completing the square lead to an equation of the form

$$y_1^2 = x^3 + \alpha_2'x^2 + \alpha_4'x + \alpha_6', \tag{4}$$

where $y_1 = y + \frac{\alpha_1}{2}x + \frac{\alpha_3}{2}$ with some constants $\alpha_2', \alpha_4', \alpha_6' \in \mathbb{K}$. In the case when the characteristic is not 2 or 3, then by substituting $x_1 = x + \frac{\alpha_2'}{3}$ in (4) we get the form

$$y_1^2 = x_1^3 + Ax_1 + B,$$

where $A, B \in \mathbb{K}$. This latter equation is referred to as Weierstrass equation for an elliptic curve. Moreover, two elliptic curves over $\mathbb{Q}$ of the form (4) that can be obtained from each others by changing of the coordinates $x = a^2X + b$ and $y = a^3Y + cX + d$, where $a, b, c, d \in \mathbb{Q}$ and then dividing by a^6 are called isomorphic. Every elliptic curve over $\mathbb{Q}$ is isomorphic to a curve of the form (4) in which the coefficients are rational integers. For deep details about the study of elliptic curves and their applications, one can see e.g. [**218**], [**256**] and the references given there.

On the other hand, if Ψ denotes the complex upper half plane. An elliptic curve $\mathbb{E}$ is called modular if there exists a homomorphism from the classical modular curve, $X_0(N)$ for a natural number $\mathbb{N}$, onto the curve $\mathbb{E}$ such that $X_0(N)$ is a compact Riemann surface that is formed by completing the noncompact quotient space $\Psi/\Gamma_0(N)$ to be compact by adjoining finitely many equivalent classes of $\mathbb{Q} \cup \{\infty\}$, called the cusps, under the act of the group

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \in SL_2(\mathbb{Z}), \text{ where } \mathbb{N} \text{ divides } a_3 \right\}.$$

Furthermore, an elliptic curve $\mathbb{E}$ over $\mathbb{Q}$ is called semistable if it is semistable at every prime number p, which means it is isomorphic to an elliptic curve over $\mathbb{Q}$ in which modulo every prime p is either nonsingular or has a singular point with two tangents of distinct directions. For more details about the modularity and semistability of elliptic curves, see e.g. [**199**], [**212**] and the references given there.

In fact, the relation between elliptic curves and Fermat's last theorem started when Taniyama $(1927 - 1958)$ proposed a problem related to the following theorem, that is called Taniyama-Shimura Conjecture (see e.g. [**211**]).

THEOREM 1.4. *Every elliptic curve over* $\mathbb{Q}$ *is modular.*

This conjecture became more clear after some related studies by Shimura. Then, in the the beginning of $1970's$ Hellegouarch [**112, 113**] connected Fermat equations of the form (1) with elliptic curves of the form

$$Y^2 = X(X + x^n)(X - y^n) \tag{5}$$

to prove some results related to elliptic curves using the results of Fermat's last theorem. On the other hand, in 1985 Frey [**85**] stated that elliptic curves arising from counterexamples to Fermat's last theorem could not be modular, and not so long later by using some ideas of Serre in [**209**] this was proved by Ribet [**196**] since he proved that if $n > 3$ is a prime number with the nonzero integers x, y and z satisfying $x^n + y^n = z^n$, then the elliptic curve (5) is not modular. Finally, Wiles presented the following theorem that leads to a proof of Fermat's last theorem.

THEOREM 1.5. *Let the nonzero integers* A_1 *and* B_1 *be relatively prime such that* $16|A_1B_2(A_1 - B_2)$*, then the elliptic curve*

$$Y^2 = X(X + A_1)(X + B_1)$$

is modular.

It is clear that the conditions of this theorem are satisfied if we let $A_1 = x^n$ and $B_1 = -y^n$, where x, y, z and n are presumptive solution of the Fermat's equation with $n > 4$ and one of the integers x, y or z is even. Thus, Theorem 1.5 and Ribet's result together imply the Fermat's last theorem. Indeed, Theorem 1.5 is proved due to his following result with Taylor:

THEOREM 1.6. *Every semistable elliptic curve over* $\mathbb{Q}$ *is modular.*

As a conclusion, having the elliptic curves (5) be modular, contradicting the conclusion of Ribet's work and establishing that counterexamples to Fermat's last theorem do not exist. For more details about the proofs of the latter two theorems and the history of Wiles' proof related to Fermat's last theorem, see e.g. [**199**], [**257**] and the references given there.

In the next section, we recall some special types of Diophantine equations with their related results, that will appear briefly or in details later with our main results.

1.2. Special types of Diophantine equations

1.2.1. Pell type Diophantine equations.

Consider the equation

$$x^2 - ay^2 = b, \tag{6}$$

where x and y are unknown rational integers, b is any rational integer and a is a given positive integer that is not a perfect square, since otherwise there are clearly no nontrivial integral solutions. Furthermore, if a is a square, then equation (6) can be written as $(x + ay)(x - ay) = b$ and therefore solved without using any further theory. Around 1768 Lagrange ([**137**], Oeuvres II, pages $377 - 535$) studied the solutions of this equation in which $\gcd(x, y) = 1$ and gave a recursive method for the general solution. This method depends on reducing this problem to the case where $|b| < \sqrt{a}$ and the solutions (x, y) are found in correspondence to (p_n, q_n) such that the fraction $(\frac{p_n}{q_n})$ converges to the simple continued fraction of $\sqrt{a}$. If $b \neq 1$, then equation (6) may have no solutions. Indeed, if there is a nontrivial solution, then it has infinitely many integer solutions (x, y) generated by a finite number of bases. Moreover, the first nontrivial solution (x_0, y_0) is called a fundamental solution of this equation. As particular equations represented in (6) are the equations $x^2 - ay^2 = \pm 1$, which are known as the classical Pell equations. The equation

$$x^2 - ay^2 = 1 \tag{7}$$

is called the Pell equation due to a mistake on the part of Euler $(1707 - 1783)$ who attributed the solution of the equation to John Pell. This equation is also known as the Pellian equation. In fact, John Pell $(1611 - 1685)$ did not make any independent contribution to this equation other than referring to it in a paper that he wrote in algebra. This equation was firstly stated by Fermat $(1601 - 1665)$ by claiming this equation has infinitely many solutions, but he did not provide a proof to it. Around the same year of announcing this problem by Fermat, Brouncker gave a systematic method for solving this equation. Lagrange [**137**] (also see e.g. [**73**, page 358]) gave a complete discussion about equation (7) with publishing a complete such a proof since he used the simple continued fraction expansion of $\sqrt{a}$ to its solvability in nontrivial rational integers x and y. It is well known that equation (7) has infinitely many solutions (u, v) given by

$$(u_k + v_k\sqrt{a}) = (u_1 + v_1\sqrt{a})^k,$$

where k is any rational integer and (u_1, v_1) is its fundamental solution. On the other hand, the equation

$$x^2 - ay^2 = -1 \tag{8}$$

is called the negative Pell equation, which does not always have a solution. Indeed, it has a solution if and only if the length of the period of the continued fraction of $\sqrt{a}$ is congruent to 1 modulo 2, see e.g. [**217**]. It is clear to observe that if (a_1, b_1) is a fundamental solution for (8), then all the solutions are given by odd powers k.

Furthermore, if the negative Pell equation (8) is solvable with the fundamental solution (a_1, b_1), then the fundamental solution of the positive Pell equation (7) can be obtained by

$$(u_1 + v_1\sqrt{a}) = (a_1 + b_1\sqrt{a})^2.$$

On the other hand, if (u_k, v_k) represents the k^{th} solution of the Pellian equation (7), then the solutions of equation (6) are given by

$$(x_k \pm y_k\sqrt{a}) = (x_b \pm y_b\sqrt{a})(u_k \pm v_k\sqrt{a}),$$

where (x_b, y_b) is a base solution of equation (6). For further details about the solutions of equation (6), see e.g. [**41**]. The problems of the determination of a fundamental solution of the equation (6) in general and deciding whether the equation (8) has a solution seem ancient problems, and they have been studied by many authors. For instance, a fundamental solution can be determined using the Chakravala method in [**78**], that was introduced by the Indian mathematicians Jayadeva (9th century) and Bhāskara II (12th century). In 1770, by using the simple continued fraction, Lagrange ([**137**], Oeuvres II, pages 655 – 726) gave one more algorithm that was considered as a generalization of the method of solving equations (7) and (8). This latter algorithm has been simplified by Matthews [**165**] and Mollin [**172**] using simple continued fractions. Kaplan and Williams [**126**] showed that the solvability of equation (8) leads to the solvability of equation (6), where $b = -4$ in positive relatively prime rational integers if and only if the length of the period of the continued fraction expansion of $\sqrt{a}$ is congruent to the length of the period of the continued fraction expansion of $(\frac{1}{2}(1+\sqrt{a}))$ modulo 4. Moreover, several authors provided other methods for finding the fundamental solutions, studying whether equation (8) has a solution or not and investigating the solvability of equation (6) in general or some specific types of this equation, see e.g. [**118**], [**146**], [**173**], [**202**], [**222**], [**240**] and the references given there.

1.2.2. Fermat-Catalan type Diophantine equations.

In 1785, Legendre [**145**] investigated the equation

$$0 = f(x, y, z) := Ax^2 + By^2 + Cz^2, \tag{9}$$

where the coefficients A, B and C are rational integers. By reducing this equation into its normal form in which these coefficients are squarefree and pairwise relatively prime, he found necessary conditions to show that this equation can be solvable in nontrivial rational integers x, y and z and stated his result in the following theorem, that was named after him.

THEOREM 1.7. *The equation* (9) *has a solution in rational integers x, y, z, not all zero, if and only if*

- *the coefficients A, B, C are not all of the same sign,*

- *there exist rational integers X, Y and Z such that $BX^2 + C \equiv 0 \pmod{A}$, $CY^2 + A \equiv 0 \pmod{B}$ and $AZ^2 + B \equiv 0 \pmod{C}$.*

Holzer [**117**] proved that a nontrivial rational integer solution exists for equation (9) under all of the above assumptions, where

$$|x| \leq \sqrt{|BC|}, \qquad |y| \leq \sqrt{|AC|}, \qquad |z| \leq \sqrt{|AB|}. \tag{10}$$

Moreover, a proof of an estimate weaker than that of Holzer's in (10) under the same assumptions was given by Mordell [**174**]. This was also found later by Skolem [**220**]. If equation (9) is solvable under the conditions of Theorem 1.7, Birch and Davenport [**30**] proved a theorem that gives the following estimate:

$$0 < |A|\, x^2 + |B|\, y^2 + |C|\, z^2 \leq 8\, |ABC|\, .$$

After that, Kneser [**133**] established a deep result of the form

$$|z| \leq k(r)\sqrt{|AB|},$$

where $r = \operatorname{lcm}(2, ABC)$ and $k(r) < 1$ in some cases. Mordell [**177**] again established an elementary proof of Holzer's estimate in (10) under the above assumptions since he showed that if a solution (x_0, y_0, z_0) exists such that x_0 and y_0 are relatively prime and $|z_0| > \sqrt{|AB|}$, then it is possible to find another solution (x, y, z) with $|z| < |z_0|$. Hence, the estimate in (10) follows. In fact, this argument is not quite complete since he did not prove the constructed integer z is nonzero. Therefore, Williams [**258**] tried to complete Mordell's proof with removing unnecessary assumptions in which A, B, C being squarefree and pairwise relatively prime by proving the following result:

THEOREM 1.8. *If the equation* (9) *has a solution in rational integers x, y, z, not all zero, where A, B, C are nonzero integers, then there exists a solution of equation* (9) *satisfying*

$$|x| \leq \frac{\sqrt{|BC|}}{(A, B, C)}, \qquad |y| \leq \frac{\sqrt{|AC|}}{(A, B, C)}, \qquad |z| \leq \frac{\sqrt{|AB|}}{(A, B, C)}.$$

On the other hand, due to Smith's observation [**223**] about carrying the conditions in Theorem 1.7 over the case in which the coefficients and unknowns are Gaussian integers, Samet [**201**] proved a similar theorem of Theorem 1.7 in $\mathbb{Z}[i]$. Furthermore, Leal-Ruperto [**143**] adapted the proof of Holzer's estimate that was established by Mordell in [**177**] to prove that if equation (9) in $\mathbb{Z}[i]$, reduced into its normal form, then it has a solution (x, y, z) which satisfies

$$|z| \leq \sqrt{(1 + \sqrt{2})\, |AB|}.$$

The latter bound was improved by Santos and Diaz-Vargas [**72**] by giving a modification of Leal-Ruperto's result [**143**] where the coefficients and unknowns are in the ring of integers of $\mathbb{Q}(\sqrt{d})$ for $d = 1, 2, 3, 7, 11$. In fact, they proved that if equation

(9), in its normal form, has a solution in $\mathbb{Q}(\sqrt{d})$ for $d = 1, 2, 3, 7, 11$ then it has a solution with

$$|z_0| \leq \sqrt{\frac{4}{3-d}|AB|} \quad \text{for} \quad d = 1, 2,$$
$$|z_0| \leq \sqrt{\frac{16d}{-d^2 + 14d - 1}|AB|} \quad \text{for} \quad d = 3, 7, 11.$$

If we consider equation (9) with $A = B = C = 1$ and $f(x, y, z) = 3xyz$, then we obtain one of the well known Diophantine equations called Markoff equation; that is,

$$x^2 + y^2 + z^2 = 3xyz,$$

which was deeply studied by Markoff $(1879, 1880)$ [**160, 161**] in case of $x, y, z \in \mathbb{Z}$ and $1 \leq x \leq y \leq z$. A triple (x, y, z) of positive integers that satisfies Markoff equation is called a Markoff triple, and the numbers x, y and z are called Markoff numbers. Markoff showed that there are infinitely many Markoff triples, which can be constructed from one fundamental solution $(1, 1, 1)$. In fact, he gave a procedure to construct new solutions from old ones. In these papers, Markoff numbers have been introduced to describe minimal values of indefinite quadratic forms with exceptionally large minima greater than $1/3$ of the square root of the discriminant. He showed that these forms are in one-to-one correspondence with the Markoff triples. More precisely, these numbers of which the first few are

$$1, 2, 5, 13, 29, 34, 89, 169, 194, 233, 433, 610, 985, 1325, 1597, \ldots$$

play a role in a famous theorem of Markoff: the $GL_2(\mathbb{Z})$-equivalence classes of real indefinite binary quadratic forms Q of discriminant 1 for which the invariant

$$\mu(Q) = \{\min|Q(x, y)| : (x, y) \in \mathbb{Z}^2 \text{ and } (x, y) \neq (0, 0)\} > \frac{1}{3}$$

are in one-to-one correspondence with the Markoff triples. Indeed, the invariant $\mu(Q)$ for the form, that is corresponding to (x, y, z) being $(9-4z^{-2})^{-1/2}$. Thus, the Markoff numbers exactly describe the part of the Markoff spectrum (the set of all $\mu(Q)$) lying above $1/3$. An equivalent result is as follows, under the action of $SL_2(\mathbb{Z})$ on $\mathbb{R} \cup \infty$ given by $t \to (at + b)/(ct + d)$, the $SL_2(\mathbb{Z})$-equivalence classes of real numbers t for which the approximation measure

$$\mu(t) = \lim_{y\to\infty} \sup \left(y \cdot \min_{x\in\mathbb{Z}} |yt - x| \right) > \frac{1}{3}$$

are in one-to-one correspondence with the Markoff triples. Indeed, the spectrum is being the same as above, namely $\mu(t) = 5^{-1/2}$ with t is equivalent to the golden ratio and $\mu(t) \leq 8^{-1/2}$ for all other t (for more details about these equivalent results, see also [**264**]). Since then, the Markoff numbers are important in both the theory of quadratic forms and in the theory of Diophantine approximation. In 1913, Frobenius [**86**] gave the following long-standing conjecture:

CONJECTURE 1.9. *If* (x, y, z) *and* (x', y', z) *are two Markoff triples, where* $x \leq y \leq z$ *and* $x' \leq y' \leq z$*, then* $x = x'$ *and* $y = y'$*.*

This conjecture is called the Markoff numbers unicity conjecture or the Frobenius unicity conjecture and has remained open. The first partial results on this conjecture were settled by Baragar [**14**] for prime Markoff numbers and by Button [**49**] and Schmutz [**207**] for Markoff numbers that are prime powers or two times a prime power. For other results related to this conjecture, see e.g. [**53**], [**138**], [**229**] and the references given there. Other generalizations and studies related to the Markoff equation will be mentioned later with our results concerning this equation in Section 3.2 of Chapter 3.

One of the interesting results related to the Legendre's equation (9) was introduced in 1844 [**52**] by the French and Belgian mathematician Catalan who wrote to Crelle's Journal, a mathematics journal that was founded by the German mathematician Crelle in 1826, the following conjecture:

CONJECTURE 1.10. *The equation*

$$x^n = y^m + 1 \tag{11}$$

has no nontrivial rational integer solutions other than $(x, y, n, m) = (\pm 3, 2, 2, 3)$ *if* $n, m > 1$.

Following this unproved conjecture, Lebesgue [**144**] proved that the equation

$$x^p = y^2 + 1$$

has no nontrivial solutions in rational integers x, y and p if p is a prime number. Cassels [**51**] proved that if equation (11) is satisfied, then $n|x$ and $m|y$, where n and m are prime numbers. On the other hand, Ko [**134**] showed that if the prime number $q \geq 5$, then equation

$$x^2 = y^q + 1$$

has no nontrivial solutions x and y in rational integers. Tijdeman [**247**] used Baker's theory in [**12**], dealing with linear forms in the logarithms of algebraic numbers, to show that there is an effective bound on the size of n and m that satisfy equation (11) such that n and m are prime numbers. There were several mathematicians who intended to prove Catalan's problem using computer technologies, but without any success till 2004 when the Romanian mathematician Mihăilescu [**170**] gave a complete proof to it using the theory of cyclotomic fields. Indeed, many developments have occurred on Catalan's conjecture, for example the Diophantine equation

$$x^2 + T = y^m, \tag{12}$$

where x, y, T, m are positive integers with $m > 2$, has been considered and studied in several papers. For instance, Cohn [**59**] studied the integral solutions x, y, m of equation (12) for several values of T in the range $1 \leq T \leq 100$. Bugeaud, Mignotte and Siksek [**48**] used some modular methods to study the solutions of (12) in the integers x, y, m, and they solved it completely for all values of T between 1 and 100. Tengely

[**242**] computed the solutions of the equations $x^2+a^2 = y^p$ and $x^2+a^2 = 2y^p$ in rational integers x, y and a prime number $p > 2$ for all a in the range $3 \le a \le 501$. Berczes and Pink [**20**] determined all the solutions of equation (12) in integers $x > 0, y > 1, m, k$ in case of $T = p^{2k}$, where k is a nonnegative integer, p is a prime number in the range of $2 \le p < 100$ and the integers x and y are relatively prime. Soydan [**224**] studied equation (12) in case of $T = 7^{a_1}11^{a_2}$ and computed all of its the solutions in nonnegative integers $x, y, a_1, a_2, m \ge 3$, where $\gcd(x, y) = 1$ except a_1x and a_2 are odd and even, respectively. For the complete history of Catalan's conjecture and other related developments, see e.g. [**38**], [**168**], [**192**], [**208**] and the references given there.

One of the most well known generalizations of Fermat's equation, Legendre's equation and Catalan's equation is the following equation, that is known as Fermat-Catalan Diophantine equation. The equation

$$0 = g(x, y, z) := A_1x^a + B_1y^b + C_1z^r \tag{13}$$

in the integers x, y, z, a, b, r, where a, b and r are greater than 1, and A_1, B_1 and C_1 are given integers such that $A_1B_1C_1 \ne 0$. Next, we mention some of the known results related to this equation, starting with the result of Darmon and Granville [**67**] in which they proved the following:

THEOREM 1.11. *For any given integers A_1, B_1, C_1, a, b, r such that $A_1B_1C_1 \ne 0$ and a, b and r greater than 1 satisfying $\frac{1}{a} + \frac{1}{b} + \frac{1}{r} < 1$, equation* (13) *has only finitely many relatively prime solutions (x, y, z) in integers.*

On the other hand, Beukers [**22**] gave the following theorem:

THEOREM 1.12. *For any given integers A_1, B_1, C_1, a, b, r such that $A_1B_1C_1 \ne 0$ and a, b and r greater than 1 satisfying $\frac{1}{a} + \frac{1}{b} + \frac{1}{r} > 1$, equation* (13) *has either no solution or infinitely many relatively prime integer solutions (x, y, z).*

For the remaining case where $\frac{1}{a} + \frac{1}{b} + \frac{1}{r} = 1$, we see an easy calculation gives that the set $\{a, b, r\}$ equals to $\{2, 4, 4\}, \{3, 3, 3\}$ or $\{2, 3, 6\}$. In this case the solution of the equation comes down to the determination of rational points on twists of genus 1 curves over $\mathbb{Q}$.

Note that if $a = b = 2, C_1 = -1$ and r is odd, then according to Mordell [**176**, page 111] one can obtain for the equation

$$A_1x^2 + B_1y^2 = z^r \tag{14}$$

the following parametrizations for its solutions by putting

$$z = A_1p^2 + B_1q^2,$$

where p and q are arbitrary integers, and taking

$$x\sqrt{A_1} + y\sqrt{-B_1} = (p\sqrt{A_1} + q\sqrt{-B_1})^r,$$
$$x\sqrt{A_1} - y\sqrt{-B_1} = (p\sqrt{A_1} - q\sqrt{-B_1})^r.$$

Therefore, x and y are expressed as polynomials in p and q. For other developments, generalizations and explicit methods related to Fermat-Catalan equation, see e.g. [**18**], [**24**], [**79**], [**84**], [**92**] and the references given there.

1.2.3. Runge type Diophantine equations.

Let $G(x,y) = 0$ be a binary Diophantine equation that is irreducible in a class including those polynomials in which the leading form of G is not a constant multiple of a power of an irreducible polynomial. In 1887, Runge [**200**] proved that such an equation $G(x,y) = 0$ has finitely many solutions. Indeed, from the proof of his method he showed the existence of an effective upper bound for each of the values of $|x|$ and $|y|$, where x and y represent integral solutions of this equation. In the following we consider one of the most well studied equation of this form, which is the polynomial

$$G(x,y) = \sum_{i=0}^{m}\sum_{j=0}^{n} b_{i,j}x^i y^j$$

of rational integer coefficients with $\deg_x(G) = m > 0$ and $\deg_y(G) = n > 0$. If $G(x,y)$ is an irreducible polynomial over $\mathbb{Q}[x,y]$, unless the following conditions hold for G, then it is said that G satisfies Runge's condition:

- for all nonzero i and j, $b_{i,n} = b_{m,j} = 0$,
- all the pairs (i,j) in the terms $b_{i,j}x^iy^j$ of the polynomial G satisfy $mn \geq ni + mj$,
- for which $nm = ni + mj$ the sum of all $b_{i,j}x^iy^j$ of G is a constant multiple of a power of an irreducible polynomial in $\mathbb{Z}[x,y]$,
- the algebraic function $y = y(x)$ defined by $G(x,y) = 0$ has only one system of conjugate Puiseux expansions at $x = \infty$.

Indeed, Runge proved that the equation $G(x,y) = 0$ has finitely many integral solutions x and y if at least one of the above conditions does not hold. The first effective upper bound of this method was obtained by Hilliker and Straus [**114**] since they used a quantitative version of Eisenstein's theorem on power series expansions of algebraic functions to show that if $M_1 = \max\{m,n\}$ and $M_2 = \max_{i,j}|a_{i,j}|$, then the solutions of $G(x,y) = 0$ satisfy

$$\max\{|x|,|y|\} < \begin{cases} 4(1+M_2)^2 & \text{if } M_1 = 1, \\ (8M_1M_2)^{M_1^{2M_1^3}} & \text{if } M_1 > 1. \end{cases}$$

Walsh [**253**] improved the result of Hilliker and Straus by using the result of Dwork and van der Poorten in [**77**] to Runge's method. In fact, he modified the result of Hilliker and Straus by removing of the double exponential in M_1 with the following result:

THEOREM 1.13. *If $G(x,y)$ satisfies Runge's Condition, then all solutions $(x,y) \in \mathbb{Z}^2$ of the Diophantine equation $G(x,y) = 0$ satisfy*

$$\max\{|x|,|y|\} < (2M_1)^{18M_1^7} M_2^{12M_1^6}.$$

Applying the Skolem's method in [**219**] based on using the elimination theory, upper bounds for the integral solutions of $G(x,y) = 0$ were obtained by Grytczuk and Schinzel [**97**]. Using the transcendental construction by algebraic functions, Laurent and Poulakis [**142**] extended Walsh's result, which was applied on the field of rational numbers, to the algebraic number fields in which they obtained an effective version of Runge's result in the context of interpolation determinants. Unfortunately, all of these bounds, that have been obtained are usually large to examine all the possibilities for the integral solutions of $G(x,y) = 0$. Beukers and Tengely [**23**] suggested a practical algorithm for determining the solutions if the coefficients and $\deg_x(G)$ and $\deg_y(G)$ are not too large. Szalay [**237**] extended the version of Runge's method given by Poulakis in [**189**] who provided a method to solve an equation of the form $y^2 = g(x)$, where $g(x)$ is a degree four polynomial with integer coefficients and nonzero discriminant, to solve completely certain Diophantine equations of the form

$$H(x,y) = z^2$$

in integers x, y and z such that

$$H(x,y) = \sum_{i+j\le 4} b_{i,j} x^i y^j$$

is a nonhomogeneous polynomial with rational integer coefficients and satisfies some technical conditions. Furthermore, many refinements, developments and generalizations related to Runge's idea have been presented by several authors, see e.g. [**28**], [**102**], [**234**], [**235**], [**241**] and the references given there.

1.2.4. Thue and superelliptic type Diophantine equations.

One of the most known general results in the theory of Diophantine equation was obtained in 1909 by Thue [**246**] who proved that the equation

$$F(x,y) = m_1, \tag{15}$$

where $F(x,y) \in \mathbb{Z}[x,y]$ is an irreducible homogeneous polynomial of degree $d > 2$ and m_1 is a nonzero rational integer, admits at most a finite number of solutions in rational integers x and y. This is called Thue's theorem.

On the other hand, the Diophantine equation

$$F(x) := a_0x^n + a_1x^{n-1} + \ldots + a_n = y^{m_2} \quad \text{in } x, y \in \mathbb{Z} \tag{16}$$

is called hyperelliptic equation if $m_2 = 2$, otherwise it is called superelliptic equation. In 1926, Siegel [**215**] proved that if $m_2 = 2$ and $F(x)$ has at least three simple roots, then equation (16) has only a finite number of solutions in rational integers. Furthermore, he showed using the same method that if $m_2 > 2$ and $F(x)$ has at least two simple roots, then equation (16) has a finite number of rational integer solutions.

Moreover, if $F(x, y)$ is an absolutely irreducible polynomial with rational integer coefficients such that $F(x, y) = 0$ represents a curve, which has a component of genus 0, Siegel [**216**] showed this equation has infinitely many integral solutions. In other words, if the algebraic curve defined by (16) is of genus > 0, then the number of the integral points on that curve is finite. We refer to this result by the Thue-Siegel's theorem. The latter Siegel's result was extended by Mahler [**159**] in which he conjectured that a similar statement holds for points having only a finite number of primes in their denominators and proved this conjecture for curves of genus 1 over the rationals $\mathbb{Q}$ by his p-adic analogue of the Thue-Siegel's theorem. Independently, Lang [**139**] and LeVeque [**147**] established the p-adic analogue of this theorem to prove that equation (16) has only finitely many S-integral solutions. Furthermore, in order to have the algebraic curve defined by equation (16) with positive genus, LeVeque [**148**] gave a necessary and sufficient condition for that. All of these results are based on Thue's method, and hence are ineffective since their proofs do not provide any algorithm for finding the solutions.

On the other hand, in the $1960's$ Baker [**7, 10**] gave a nontrivial effective lower bound for the linear forms of logarithms

$$\Lambda = b_1 \log a_1 + b_2 \log a_2 + \ldots + b_m \log a_m,$$

where $m \in \mathbb{Z}, m > 0, b_i \in \mathbb{Z}$ for $i = 1, \ldots, m$ and $a_1, a_2, \ldots, a_m$ are any algebraic numbers that are not 0 or 1, and $\log$ denotes fixed determination of the logarithmic function. Therefore, by using his estimates, Baker [**8**] gave an effective proof of Thue's theorem and provided an algorithm for the complete solutions of (15), which is stated in the following theorem:

THEOREM 1.14. *All the integral solutions x and y of the Thue equation* (15)*, under its conditions with $d < C_2 - 1$, satisfy*

$$\max(|x|, |y|) < C_1 \exp\{(\log m_1)^{C_2}\},$$

where C_1 is computed effectively depending only on the values of d, C_2 and the coefficients of $F(x, y)$.

As applications of this method, Baker [**9, 11**] extended the result that he established on the Diophantine equation

$$y^2 = ax^3 + bx^2 + cx + d$$

to the class of the Diophantine equations of the form (16) as stated in the following theorems:

THEOREM 1.15. *If $m_2 > 2$, $n > 2$, $a_0 \neq 0, a_1, a_2, \ldots, a_n \in \mathbb{Z}$, and the polynomial on the left hand side of the superelliptic equation of* (16) *has at least two simple roots,* *then all the integral solutions x and y of equation* (16) *satisfy*

$$\max(|x|,|y|) < \exp\exp\{(5m_2)^{10}(n^{10n}H)^{n^2}\},$$

where $H = \{\max | a_j |: \textit{for all } j = 0, \ldots, n\}$.

THEOREM 1.16. *If* $m_2 = 2$ *and the polynomial on the left hand side of the hyper-elliptic equation of* (16) *has at least three simple roots, then all the integral solutions* x *and* y *of equation* (16) *satisfy*

$$\max(|x|,|y|) < \exp\exp\exp\{(n^{10n}H)^{n^2}\}.$$

In the $1970's$ Baker's estimates were improved by Sprindžuk [**225, 226, 227**] whose results were also generalized to the S-integral solutions of (16) by Trelina [**248**]. Using the mentioned results of Baker and LeVeque; Györy, Tijdeman and Voorhoeve [**100, 101**] proved finiteness results concerning some exponential Diophantine equations. But, due to the ineffective character of LeVeque's theorem, their earlier result in [**100**] is ineffective. Five years later, Brindza [**37**] improved the latter result to be effective. Moreover, Baker's bounds have also been improved and generalized by several authors, including Shorey and Tijdeman [**214**], Bugeaud [**42**], Hajdu and Herendi [**103**]. In fact, the best known bounds concerning the solutions of elliptic equations over $\mathbb{Q}$ are due to Hajdu and Herendi [**103**] in 1998. For other improvements and applications of Baker's method on other types of Diophantine equations, see e.g. [**25**], [**29**], [**99**], [**188**], [**206**], [**228**], [**251**] and the references given there.

In the following section, we recall some important concepts and notations related to the subject of linear recurrence sequences, that will be used throughout the dissertation. Then we recite some recent results related to the solutions of some Diophantine equations concerning particular linear recurrence sequences.

1.3. Diophantine properties of linear recursive sequences

1.3.1. Background and notations.

The linear recurrences have an ancient history in Number Theory especially in the study of particular Diophantine equations. In fact, they have been widely studied for their own sake and also as auxiliary tools toward other Diophantine problems. A sequence $(G_n)_{n\geq 0} \subseteq \mathbb{C}$ (it is also denoted by $\{G_n\}_{n\geq 0}$ or simply $\{G_n\}$) is called a linear recurrence relation of order k if the recurrence

$$G_{n+k} = a_1 G_{n+k-1} + a_2 G_{n+k-2} + \ldots + a_k G_n + f(n)$$

holds for all $n \geq 0$ with the coefficients $a_1, a_2, \ldots, (a_k \neq 0) \in \mathbb{C}$ and $f(n)$ a function depending on n only. If $f(n) = 0$ such a recurrence relation is called homogeneous, otherwise it is called nonhomogeneous.

For the homogeneous recurrence relation, the polynomial

$$F(X) = X^k - a_1X^{k-1} - \ldots - a_k = \prod_{i=1}^{s}(X-\alpha_i)^{r_i} \in \mathbb{C}[X],$$

where $\alpha_1, \alpha_2, \ldots, \alpha_s$ and $r_1, r_2, ..., r_s$ are respectively the distinct roots of $F(X)$ and their corresponding multiplicities, is called the characteristic polynomial of $(G_n)_{n\geq 0}$. Thus, if $F(X) \in \mathbb{Z}[X]$ has k distinct roots, then there exist constants $c_1, c_2, \ldots, c_k \in \mathbb{Q}(\alpha_1, \alpha_2, \ldots, \alpha_k)$ such that

$$G_n = \sum_{i=1}^{k} c_i\alpha_i^n$$

holds for all the nonnegative values of n. This sequence is called a non-degenerate sequence if none of the quotients α_i/α_j $(1 \leq i < j \leq s)$ is a root of unity. If $k = 3$, then the sequence is called a ternary linear recurrence sequence. Most of the well known ternary linear recurrence sequences are the Tribonacci sequence and Berstel's sequence, that are given by

$$T_0 = T_1 = 0, T_2 = 1, \quad T_{n+3} = T_{n+2} + T_{n+1} + T_n \qquad \text{for} \quad n \geq 0, \tag{17}$$

$$B_0 = B_1 = 0, B_2 = 1, \quad B_{n+3} = 2B_{n+2} - 4B_{n+1} + 4B_n \qquad \text{for} \quad n \geq 0, \tag{18}$$

respectively. On the other hand, if $k = 2$ and a_1 and a_2 are nonzero integers such that $a_1^2 + 4a_2 \neq 0$, then $(G_n)_{n\geq 0}$ represents a binary recurrence sequence whose characteristic polynomial is

$$F(X) = X^2 - a_1X - a_2 = (X-\alpha_1)(X-\alpha_2),$$

with $\alpha_1 \neq \alpha_2$. Again, this sequence is called a non-degenerate sequence if $c_1c_2\alpha_1\alpha_2 \neq 0$ and the ratio $\frac{\alpha_1}{\alpha_2}$ is not a root of unity. Since a major part of our main results will be mainly depending on considering special non-degenerate binary linear recurrence sequences, let us define these sequences in a more precise and detailed way. Let P and Q be nonzero relatively prime integers and $D = P^2 - 4Q$ be called the discriminant. Let $U_n = U_n(P,Q)$ and $V_n = V_n(P,Q)$ be defined by the following recurrence relations with their initials:

$$U_0 = 0, U_1 = 1, \quad U_n = PU_{n-1} - QU_{n-2} \quad \text{for} \quad n \geq 2, \tag{19}$$

$$V_0 = 2, V_1 = P, \quad V_n = PV_{n-1} - QV_{n-2} \quad \text{for} \quad n \geq 2, \tag{20}$$

respectively. Therefore, the characteristic polynomial of the recurrences is given by

$$X^2 - PX + Q,$$

which has the roots

$$\alpha = \frac{P+\sqrt{D}}{2} \quad \text{and} \quad \beta = \frac{P-\sqrt{D}}{2},$$

with $\alpha \neq \beta, \alpha + \beta = P, \alpha \cdot \beta = Q$ and $(\alpha - \beta)^2 = D$. The sequences $\{U_n\}$ and $\{V_n\}$ are called the (first and second kind) Lucas sequences with the parameters (P,Q), respectively. Moreover, $\{U_n\}$ is also called Lucas sequence and $\{V_n\}$ is called the associated Lucas sequence or the companion Lucas sequence. Sometimes these sequences are both called Lucas sequences, and the numbers in them are the generalized Lucas numbers. The terms of the Lucas sequences of the first and second kind satisfy the identity

$$V_n^2 = DU_n^2 + 4Q^n. \tag{21}$$

Furthermore, the Lucas sequences of the first and second kind can be respectively written by the following formulas that are known as Binet's formulas:

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad V_n = \alpha^n + \beta^n \quad \text{for} \quad n \geq 0. \tag{22}$$

As mentioned earlier, if the ratio $\zeta = \frac{\alpha}{\beta}$ is a root of unity, then the sequences $\{U_n\}$ and $\{V_n\}$ are said to be degenerate, and non-degenerate otherwise. Describing all of the degenerate Lucas sequences of first and second kind follows from the fact that

$$\left|\zeta + \zeta^{-1}\right| = \left|\frac{\alpha}{\beta} + \frac{\beta}{\alpha}\right| \leq 2.$$

Since $\frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{P^2-2Q}{Q}$, it follows that $P^2 - 2Q = 0, \pm Q, \pm 2Q$. This implies that $P^2 = Q$, $2Q, 3Q, 4Q$. Since $\gcd(P,Q) = 1$, we have $(P,Q) = (1,1), (-1,1), (2,1)$ or $(-2,1)$. Therefore, if $D = 0$ or $D = -3$, then the sequence is degenerate. For more details about the degenerate and non-degenerate Lucas sequences of the first and second kind and the proof of Binet's formulas, see e.g. (Chapter 1 in [**193**]). Most of the well known and interesting Lucas sequences of the first and second kind are the sequences of the Fibonacci numbers, Pell numbers, Lucas numbers, Pell-Lucas numbers, Jacobsthal numbers, Jacobsthal-Lucas numbers, balancing numbers and companion balancing numbers, which are respectively given by

$$F_0 = 0, F_1 = 1, \quad F_n = F_{n-1} + F_{n-2} \quad \text{for} \quad n \geq 2, \tag{23}$$
$$P_0 = 0, P_1 = 1, \quad P_n = 2P_{n-1} + P_{n-2} \quad \text{for} \quad n \geq 2, \tag{24}$$
$$L_0 = 2, L_1 = 1, \quad L_n = L_{n-1} + L_{n-2} \quad \text{for} \quad n \geq 2, \tag{25}$$
$$Q_0 = 2, Q_1 = 2, \quad Q_n = 2Q_{n-1} + Q_{n-2} \quad \text{for} \quad n \geq 2, \tag{26}$$
$$J_0 = 0, J_1 = 1, \quad J_n = J_{n-1} + 2J_{n-2} \quad \text{for} \quad n \geq 2, \tag{27}$$
$$j_0 = 2, j_1 = 1, \quad j_n = j_{n-1} + 2j_{n-2} \quad \text{for} \quad n \geq 2, \tag{28}$$
$$B_0 = 0, B_1 = 1, \quad B_n = 6B_{n-1} - B_{n-2} \quad \text{for} \quad n \geq 2, \tag{29}$$
$$b_0 = 2, b_1 = 6, \quad b_n = 6b_{n-1} - b_{n-2} \quad \text{for} \quad n \geq 2. \tag{30}$$

Therefore, the Binet's formulas for these eight sequences are given as follows

$$F_n = \frac{\alpha_1^n - \beta_1^n}{\alpha_1 - \beta_1} \quad \text{and} \quad L_n = \alpha_1^n + \beta_1^n \quad \text{for} \quad n \geq 0, \tag{31}$$
$$P_n = \frac{\alpha_2^n - \beta_2^n}{\alpha_2 - \beta_2} \quad \text{and} \quad Q_n = \alpha_2^n + \beta_2^n \quad \text{for} \quad n \geq 0, \tag{32}$$
$$J_n = \frac{\alpha_3^n - \beta_3^n}{\alpha_3 - \beta_3} \quad \text{and} \quad j_n = \alpha_3^n + \beta_3^n \quad \text{for} \quad n \geq 0, \tag{33}$$

$$B_n = \frac{\alpha_4^n - \beta_4^n}{\alpha_4 - \beta_4} \quad \text{and} \quad b_n = \alpha_4^n + \beta_4^n \qquad \text{for} \quad n \geq 0, \tag{34}$$

where $(\alpha_1, \beta_1) = (\frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}), (\alpha_2, \beta_2) = (1+\sqrt{2}, 1-\sqrt{2}), (\alpha_3, \beta_3) = (2, -1)$ and $(\alpha_4, \beta_4) = (3+2\sqrt{2}, 3-2\sqrt{2})$.

Furthermore, based on the above Binet's formulas one can easily show that

$$\alpha_1^{n-2} \leq F_n \leq \alpha_1^{n-1} \quad \text{and} \quad \alpha_1^{n-1} \leq L_n \leq \alpha_1^{n+1} \qquad \text{for} \quad n \geq 1, \tag{35}$$

$$\alpha_2^{n-2} \leq P_n \leq \alpha_2^{n-1} \quad \text{and} \quad \alpha_2^{n-1} \leq Q_n \leq \alpha_2^{n+1} \qquad \text{for} \quad n \geq 1, \tag{36}$$

$$\frac{\alpha_3^{n-1}}{3} \leq J_n \leq \alpha_3^{n-1} \quad \text{and} \quad \alpha_3^n - 1 \leq j_n \leq \alpha_3^n + 1 \qquad \text{for} \quad n \geq 1, \tag{37}$$

$$\alpha_4^{n-1} \leq B_n \leq \alpha_4^n \quad \text{and} \quad \alpha_4^n - 1 \leq b_n \leq \alpha_4^n + 1 \qquad \text{for} \quad n \geq 1. \tag{38}$$

Historically, the study of Lucas sequences goes back to the $XIII^{th}$ century. For instance, the Fibonacci numbers were first introduced in 1202 by the Italian mathematician Fibonacci who was also known as Leonardo of Pisa in his book Liber abaci concerning the reproduction patterns of rabbits. So the sequence of Fibonacci numbers appeared in the solution of the following rabbits' problem:

" SUPPOSE WE HAVE A PAIR OF EARLY BORN RABBITS AND AFTER THEY GET MATURED, THEY BEGET EVERY MONTH A NEW PAIR OF RABBITS THAT BECOMES PRODUCTIVE AT THE AGE OF TWO MONTHS. HOW MANY PAIRS OF RABBITS CAN BE PRODUCED IN A YEAR IF WE ASSUME THAT THE RABBITS NEVER DIE ? "

Since that time till our days many mathematicians have studied and investigated the Fibonacci sequence, which appears in several areas of life and sciences. For example, the Fibonacci sequence has many connections and applications in music, architecture, painting, chemistry, medical sciences, geography and physics; see e.g. Lowman [**150**], Preziosi [**190**], Hedian [**111**], Wlodarski [**260**], Hung, Shannon and Thornton [**121**], Sharp [**210**] and Davis [**68**], respectively.

On the other hand, the first significant work on the subject is by Lucas with his seminal paper of 1878. Subsequently, many papers appeared related to prime divisions of special sequences of binomials. A prime number p is called a primitive divisor of the Lucas number U_n if p divides U_n but it does not divide $(\alpha-\beta)^2 U_1 \dots U_{n-1}$, where α and β are the roots of the characteristic polynomial of the Lucas sequences. The first general result about the existence of the primitive divisors goes back to the early of 1892 due to the work of Zsigmondy [**265**] when he proved that for all $n > 6$, U_n has a primitive divisor in case of $\alpha, \beta \in \mathbb{Z}$. Independently, after 12 years later, this result rediscovered by Birkhoff and Vandiver [**31**]. In case of $\alpha, \beta \in \mathbb{R}$, the same result was obtained by Carmichael [**50**]. Moreover, Schinzel [**203**] proved that if $\alpha, \beta \in \mathbb{C}$ and $\frac{\alpha}{\beta}$ is not a root of unity and if n is sufficiently large, then the n^{th} term in the associated Lucas sequence has a primitive divisor. Stewart [**232**] showed that if $n = 5$ or $n > 6$, then there are only finitely many Lucas sequences that do not have a primitive divisor.

Following the prior works of Schinzel and Stewart; Bilu, Hanrot and Voutier [**26**] finished off the case when the roots $\alpha, \beta \in \mathbb{C}$ such that $\alpha, \beta \notin \mathbb{R}$.

Many of the mentioned results related to the primitive divisor of the Lucas sequence extended to a known general version of the Lucas sequence that is called a Lehmer sequence, which is defined as the following: Let a_1 and a_2 be nonzero relatively prime integers, and let α and β be the roots of the equation

$$x^2 - \sqrt{a_1}x - a_2 = 0,$$

with $\frac{\alpha}{\beta}$ is not a root of unity. Then the Lehmer sequence $(W_n)_{n\geq 0}$ is defined by

$$W_n = \begin{cases} \frac{\alpha^n - \beta^n}{\alpha - \beta} & \text{if } n \text{ is odd,} \\ \\ \frac{\alpha^n - \beta^n}{\alpha^2 - \beta^2} & \text{if } n \text{ is even.} \end{cases}$$

A prime number p is called a primitive divisor of a Lehmer number W_n if p divides W_n but it does not divide $(\alpha^2 - \beta^2)^2 W_1 \dots W_{n-1}$. There are many related results to the Lehmer sequence. For instance, Ward [**254**] extended the result of Carmichael [**50**] to the Lehmer sequence in case of $\alpha, \beta \in \mathbb{R}$. Bilu, Hanrot and Voutier [**26**] also dealt with the case where the roots α, β are complex but not real.

For other developments, generalizations and applications related to the primitive divisors of Lucas and Lehmer sequences, see e.g. [**152**], [**154**], [**252**], [**263**] and the references given there.

1.3.2. Historical survey.

Several authors have studied the solutions of different types of Diophantine equations related to linear recurrence sequences. For instance, consider the mixed exponential-polynomial Diophantine equation

$$G_n = P(x), \tag{39}$$

where G_n denotes the n^{th} (or general) term of a linear recurrence sequence and $P(x) \in \mathbb{Z}[x]$ is a polynomial of degree $d > 1$. Many authors have investigated the integral solutions (n, x) of equation (39) by obtaining an upper bound for the number of solutions, proving that the number of solutions is finite or using some techniques to solve such a problem completely. However, there are not many methods that prove the completeness of the set of the solutions of this problem.

In case of the polynomial $P(x) = 0$, several authors proved the finiteness of the number of the solutions of equation (39) such as Evertse [**80**] and Schlickewei and van der Poorten [**250**] in which they proved some related general finiteness results for such equations. Also, Schlickewei [**204**] gave an upper bound for the number of solutions of equation (39). For other results related to this case, see e.g. [**81**], [**141**], [**152**], [**205**] and the references given there.

On the other hand, the case of $P(x) \neq 0$ has been widely investigated by many mathematicians since ancient times. The following problem is one of the historical problems of the form (39) in which $P(x) \neq 0$, which was posed in 1963 by both Moser and Carlitz [**178**], and Rollett [**197**]. In the Fibonacci sequence $\{F_n\}$: "Are there any squares other than $F_0 = 0, F_1 = F_2 = 1$ and $F_{12} = 144$?" The first result was obtained by Wunderlich [**261**] using a sieving procedure to prove that the only squares among the first one million Fibonacci numbers are $0, 1$ and 144. Independently, Cohn [**56, 57**] and Wyler [**262**] used elementary methods to prove that the only squares in the sequence of Fibonacci numbers are $F_0 = 0, F_1 = F_2 = 1$ and $F_{12} = 144$. Furthermore, by using Wunderlich's technique with an upper bound for the possible solution obtained using a lower bound for linear forms in logarithms of algebraic numbers, Pethö [**183**] proved that the only cubes in the sequence of Fibonacci numbers are $F_0 = 0, F_1 = F_2 = 1$ and $F_6 = 8$. Later in [**184**], he proved that only $F_0 = 0$ and $F_1 = F_2 = 1$ are the fifth power Fibonacci numbers. Bugeaud, Mignotte and Siksek [**48**] proved that the only perfect powers in the Fibonacci sequence are $F_0 = 0, F_1 = F_2 = 1, F_6 = 8$ and $F_{12} = 144$, and the only perfect powers in the Lucas sequence are $L_1 = 1$ and $L_3 = 4$ by applying a combination of classical techniques with the modular approach to solve the following pair of equations:

$$F_n = y^p, \quad n \geq 0 \text{ and } p \text{ is prime},$$

and

$$L_n = y^p, \quad n \geq 0 \text{ and } p \text{ is prime}.$$

The former equation was generalized by Bugeaud, Luca and others [**45**] where they solved the equation $F_n \pm 1 = y^p$ completely in nonnegative integer solutions (n, y, p), where $p \geq 2$.

Using Cohn's idea in [**56, 57**], Ribenboim and McDaniel [**195**] determined all the indices n in which each of $U_n(P,Q), 2U_n(P,Q), V_n(P,Q)$ or $2V_n(P,Q)$ is a square with odd relatively prime parameters P and Q. This result was extended by Mignotte and Pethö [**169**] since they proved that if $P \geq 3$ and $Q = 1$, then the equation $U_n(P,1) = \square$ has the solutions $(P,n) = (338,4)$ or $(3,6)$ for all $n \geq 3$. In case of $P \geq 4$ and $Q = 1$, they showed that the equation $U_n(P,1) = d\square$ with $d = 2,3$ or 6 has no solution for all $n \geq 4$. Nakamula and Pethö [**179**] determined the solutions of the equation $U_n(P,-1) = d\square$ with $d = 1,2,3$ or 6. Karaatli [**127**] determined the solutions of the equation $V_n(P,Q) = 7kx^2$ for all odd relatively prime values of P and Q such that P is divisible by k.

Furthermore, when P is odd Cohn [**58**] studied the solutions of the Diophantine equations $V_n(P,Q) = V_m(P,Q)x^2$ and $V_n(P,Q) = 2V_m(P,Q)x^2$. Moreover, the Diophantine equations $F_n = 2F_mx^2$, $L_n = 2L_mx^2, F_n = 3F_mx^2, F_n = 6F_mx^2$ and $L_n = 6L_mx^2$ were solved completely by Keskin and Yosma [**130**]. Assuming that $Q = -1$ and P is odd, Şiar and Keskin [**63**] solved the equation $V_n(P,-1) = kx^2$ in case of k divides P. They also showed that the equations $V_n(P,-1) = 3x^2$ for $n \geq 3$ and $V_n(P,-1) = 6x^2$ have no solutions. Furthermore, under these assumptions they determined the solutions of the equations $V_n(P,-1) = 3V_m(P,-1)x^2$

and $V_n(P,-1) = 6V_m(P,-1)x^2$. They also found all of the solutions of the equations $V_n(P,-1) = 3x^2$ and $V_n(P,-1) = 3V_m(P,-1)x^2$ for which P is even. Keskin and Karaatli [**129**] studied the solutions of the equations $U_n(P,-1) = 5x^2$ and $U_n(P,-1) = 5U_m(P,-1)x^2$ for some assumptions on P. Indeed, if P is odd, they proved that the equation $V_n(P,-1) = 5x^2$ is solvable only when $n = 1$ while the equation $V_n(P,-1) = 5V_m(P,-1)x^2$ is unsolvable. For other results related to such problems, see e.g. [**3**], [**47**], [**54**], [**61**], [**98**], [**152**] and the references given there.

Other types of the Diophantine equations related to linear recurrence sequences were investigated by other authors. For instance, Ribenboim and McDaniel studied the following type of problems. Let $\{u_n\}$ be a second order linear recursive sequence. The terms u_n and u_m are said to be in the same square-class if there exists a nonzero integer x such that

$$u_n u_m = x^2.$$

A square-class is called nontrivial if it contains more than one term of the sequence. Otherwise, it is trivial. Ribenboim [**191**] showed that if $m \neq 1, 2, 3, 6$ or 12, then the square-class of a Fibonacci number F_m is trivial, which means the equation $F_m F_h = x^2$ has no solution in nonzero rational integers x and h such that $h \neq m$. Furthermore, he proved that the square-class of a Lucas number L_m is trivial if $m \neq 0, 1, 3$ or 6. These results were extended by Ribenboim and McDaniel for more general sequences, i.e. the first and second kind Lucas sequences. If P and Q are odd relatively prime integers and $D = P^2 - 4Q > 0$, in [**194**] they showed that each square-class of each sequence is finite, and it can be computed effectively. Also, in [**167**] they proved that there are finitely many nontrivial square-classes in $\{U_n(P,Q)\}$ and $\{V_n(P,Q)\}$ and no class contains more than three elements. In fact, they determine the integral solutions m and n of the equations $U_m(P,Q)U_n(P,Q) = x^2$ and $V_m(P,Q)V_n(P,Q) = x^2$ in case of $1 \leq m < n$ and $n \neq 3m$. Moreover, they showed that there exists a constant C such that $m < C$, which can be computed effectively in case of $n = 3m$ for $m > 1$ and $U_m(P,Q)U_n(P,Q) = x^2$ and $V_m(P,Q)V_n(P,Q) = x^2$. If $|Q+1| < P$, they proved that no additional square-classes exist when $n = 3m$. For more related results, see e.g. [**64**], [**132**], [**158**] and the references given there.

In addition to the above mentioned problems, the Diophantine triples and reduced quadruples concerning linear recurrence sequences are very important and investigated problems that have a rich history. A Diophantine m-tuple is a set $\{a_1, a_2, ..., a_m\}$ of positive integers such that $a_i a_j + 1$ is a square for all $1 \leq i < j \leq m$. Historically, after Fermat found the integer quadruple $\{1, 3, 8, 120\}$, Diophantus found the rational quadruple $\{1/16, 33/16, 17/4, 105/16\}$. Indeed, it is clear that there are infinitely many Diophantine quadruples of integers. Dujella [**75**] proved that there is no Diophantine sextuple, but there are finitely many Diophantine quintuples, which can be computed effectively. There have been some generalizations of this problem appearing in several papers. For instance, replacing the squares by higher powers of fixed or variable exponents was considered by Bugeaud and Dujella in [**43**], Bugeaud and Gyarmati in [**44**] and Luca in [**153**]; or replacing the squares by members of non-degenerate binary recurrences by Fuchs, Luca and Szalay in [**90**]. Moreover, Luca and Szalay [**156**] proved that there are no Diophantine triples (a, b, c) for which the components a, b and c are distinct positive integers with $ab + 1, ac + 1$ and $bc + 1$ present all three members of values in the Fibonacci sequence $(F_n)_{n\geq 0}$. Furthermore, they [**157**] showed that $(a, b, c) = (1, 2, 3)$ is the only Diophantine triple with values in the Lucas companion $(L_n)_{n\geq 0}$ of the Fibonacci sequence. Ruiz and Luca [**93**] generalized the latter results in case of the Tribonacci sequence $(T_n)_{n\geq 0}$. In fact, they proved that there exist no positive integers $a_1 < a_2 < a_3 < a_4$ such that $a_i a_j + 1 = T_{n_{i,j}}$ with $1 \leq i < j \leq 4$, for some positive integers $n_{i,j}$. For more related results, see e.g. [**87**], [**88**], [**238**] and the references given there.

For other types of Diophantine equation problems related to linear recurrences, see e.g. [**19**], [**21**], [**27**], [**34**], [**89**], [**131**], [**162**], [**185**],[**186**], [**187**], [**213**] and the references given there.

1.4. Summary of new results and plan of the dissertation

It is often that, with the currently available methods, we are unable to completely solve the problems of Diophantine equations concerning linear recurrence sequences. In fact, we are usually able to obtain an upper bound for the number of solutions or to prove the finiteness of the number of solutions. Therefore, in this dissertation we investigate the solutions of some well known Diophantine equations connected to some linear recurrence sequences using smooth techniques with which we show whether or not such special solutions exist and then determine the complete set of solutions. The literature of the investigated problems will be mentioned together with our main results. We also remark that regarding the notations and preliminaries of the linear recurrence sequences, that will appear latter in the following two chapters, we use the same ones presented in Section 1.3 of Chapter 1.

In Chapter 2, we respectively study the integral solutions of some Diophantine equations related to reciprocals and repdigits with linear recurrence sequences. This chapter contains two sections. In the first section, we provide results for the solutions of the following Diophantine equations:

- equations of the form
$$\frac{1}{U_n(P_2, Q_2)} = \sum_{k=1}^{\infty} \frac{U_{k-1}(P_1, Q_1)}{x^k},$$
where the sequence $\{U_n(P, Q)\}$ represents the Lucas sequence of the first kind with certain pairs (P, Q) such that $(P_1, Q_1) \neq (P_2, Q_2)$.
- equations of the form
$$\sum_{k=1}^{\infty} \frac{U_{k-1}(P, Q)}{x^k} = \sum_{k=1}^{\infty} \frac{R_{k-1}}{y^k},$$

with some pairs (P, Q), and the sequence $\{R_n\}$ is a ternary linear recurrence sequence represented by the Tribonacci sequence $\{T_n\}$ or Berstel's sequence $\{B_n\}$.

- equations of the form
$$\sum_{k=1}^{\infty} \frac{U_{k-1}(P_1, Q_1)}{x^k} = \sum_{k=1}^{\infty} \frac{U_{k-1}(P_2, Q_2)}{y^k},$$
where the pairs $(P_1, Q_1) \neq (P_2, Q_2)$.
- equations of the form
$$\sum_{k=1}^{\infty} \frac{T_{k-1}(a_2, a_1, a_0)}{x^k} = \sum_{k=1}^{\infty} \frac{T_{k-1}(b_2, b_1, b_0)}{y^k},$$
where the triples $(a_2, a_1, a_0) \neq (b_2, b_1, b_0)$ and T_n denotes the general term of the generalized Tribonacci sequence that is given by
$$T_0(p, q, r) = T_1(p, q, r) = 0, T_2(p, q, r) = 1 \quad \text{and}$$
$$T_n(p, q, r) = pT_{n-1}(p, q, r) + qT_{n-2}(p, q, r) + rT_{n-3}(p, q, r),$$
for $n \geq 3$.

In the second section, we firstly give a finiteness result for the solutions of the Diophantine equation
$$G_n = B \cdot \left(\frac{g^{lm} - 1}{g^l - 1}\right),$$
where $(G_n)_{n\geq1}$ is an integer linear recurrence sequence represented by the Lucas sequence of the first kind $\{U_n(P, Q)\}$ or of the second kind $\{V_n(P, Q)\}$ (in case of $Q = \pm1$) and $n, m, g, l, B \in \mathbb{Z}^+$ such that $m, g > 1$, l is even and $1 \leq B \leq g^l - 1$. Then we apply this result on some binary recurrence sequences, e.g. the sequences of Fibonacci numbers $\{F_n\}$ and Pell numbers $\{P_n\}$, to solve such equations completely. Furthermore, with the first application we determine all the solutions (n, m, g, B, l) of the equation $F_n = B \cdot \left(\frac{g^{lm}-1}{g^l-1}\right)$, where $2 \leq g \leq 9$ and $l = 1$.

In Chapter 3, we investigate the solutions of some Diophantine equations of the form $G(X, Y, Z) := AX^2 + BY^r + CZ^2$ (that have infinitely many solutions in rational integers) from particular linear recurrence sequences for certain nonzero integers A, B, C and r. This chapter consists of two sections in which we respectively study the solutions of such equations in case of $G(X, Y, Z) = 0$ and in case of $G(X, Y, Z) \neq 0$. In the first section, we present a technique with which we can investigate the nontrivial integer solutions (X, Y, Z) of any equation, that satisfies some conditions, of the form
$$AX^2 + Y^r = C'Z^2,$$
for certain nonzero integers A, C' and r with $r > 1$ is odd and $(X, Y) = (L_n, F_n)$ (or $(X, Y) = (F_n, L_n)$), where F_n and L_n denote the general terms of the sequences of Fibonacci numbers and Lucas numbers, respectively. More precisely, we represent the procedure of this technique in case of $(A, C', r) = (7, 1, 7)$.

In the other section, we provide a technique for studying the solutions of some generalizations of Markoff equation from certain binary linear recurrence sequences. This section contains two subsections. In the first subsection, we determine all the solutions $(X,Y,Z) = (F_I, F_J, F_K)$, where F_I, F_J and F_K represent nonzero Fibonacci numbers satisfying a generalization of the Markoff equation called the Jin-Schmidt equation; that is,

$$AX^2 + BY^2 + CZ^2 = DXYZ + 1,$$

where $(A,B,C,D) \in S$, with

$$S = \{(2,2,3,6),(2,1,2,2),(7,2,14,14),(3,1,6,6),(6,10,15,30),(5,1,5,5)\}.$$

In the other one, we consider the generalized Lucas number solutions of another generalization of the Markoff equation called the Markoff-Rosenberger equation, that has the form

$$ax^2 + by^2 + cz^2 = dxyz,$$

where $(a,b,c,d) \in T$ such that

$$T = \{(1,1,1,1),(1,1,1,3),(1,1,2,2),(1,1,2,4),(1,1,5,5),(1,2,3,6)\}.$$

In other words, we give results for the solutions $(x,y,z) = (R_i, R_j, R_k)$, where R_i denotes the i^{th} generalized Lucas number of first/second kind, i.e. $R_i = U_i$ or V_i. Furthermore, we apply the results to completely resolve concrete equations, e.g. we determine solutions containing only balancing numbers B_n and Jacobsthal numbers J_n, respectively.

The results of this dissertation have been published in the papers [**104, 105, 106, 107**] and accepted for publication in the papers [**108**, Mathematica Bohemica journal] and [**110**, Periodica Mathematica Hungarica journal]. As one can see, the main theme of our theorems is Diophantine equations involving linear recurrences sequences. We also note that we have a different result in [**109**] (that is accepted for publication in the journal "Rad HAZU, Matematičke znanosti") in which we use the frequency analysis technique to break a public key cryptosystem called ITRU, which is a variant of NTRU (N^{th} Degree Truncated Polynomial Ring) cryptosystem. However, to keep the presentation coherent, this result is not included in this dissertation.

CHAPTER 2

Diophantine equations related to reciprocals and repdigits with linear recurrence sequences

2.1. Diophantine equations related to reciprocals of linear recurrence sequences

Suppose that P and Q are nonzero relatively prime integers and $\{U_n(P,Q)\}$ and $\{V_n(P,Q)\}$ represent the Lucas sequences of the first and second kind, which are defined by (19) and (20), respectively. The study of the representations of reciprocals of Lucas sequences has been of interest to many mathematicians, and in the following we mention some of the related results. In 1953, Stancliff [**230**] noted an interesting property of the Fibonacci number $F_{11} = 89$ in the the Fibonacci sequence $\{F_n\} = \{U_n(1,-1)\}$, namely

$$\frac{1}{F_{11}} = \frac{1}{89} = 0.0112358\ldots = \sum_{k=0}^{\infty} \frac{F_k}{10^{k+1}}.$$

In 1980, Winans [**259**] studied the related sums

$$\sum_{k=0}^{\infty} \frac{F_{\alpha k}}{10^{k+1}}$$

for certain values of α. In 1981, Hudson and Winans [**120**] characterized all decimal fractions that can be approximated by sums of the type

$$\frac{1}{F_\alpha} \sum_{k=1}^{n} \frac{F_{\alpha k}}{10^{l(k+1)}}, \quad \alpha, l \geq 1.$$

In the same year, Long [**149**] obtained a general identity for binary recurrence sequences from which one obtains e.g.

$$\frac{1}{109} = \sum_{k=0}^{\infty} \frac{F_k}{(-10)^{k+1}}, \qquad \frac{1}{10099} = \sum_{k=0}^{\infty} \frac{F_k}{(-100)^{k+1}}.$$

In 1995, in case of the equation

$$\frac{1}{U_n(P,Q)} = \sum_{k=1}^{\infty} \frac{U_{k-1}(P,Q)}{x^k},$$

De Weger [**70**] determined all $x \geq 2$ in case of $(P, Q) = (1, -1)$. The solutions are as follows

$$\frac{1}{F_1} = \frac{1}{F_2} = \frac{1}{1} = \sum_{k=1}^{\infty} \frac{F_{k-1}}{2^k}, \qquad \frac{1}{F_5} = \frac{1}{5} = \sum_{k=1}^{\infty} \frac{F_{k-1}}{3^k},$$
$$\frac{1}{F_{10}} = \frac{1}{55} = \sum_{k=1}^{\infty} \frac{F_{k-1}}{8^k}, \qquad \frac{1}{F_{11}} = \frac{1}{89} = \sum_{k=1}^{\infty} \frac{F_{k-1}}{10^k}.$$

In 2009, Ohtsuka and Nakamura [**181**] proved that

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{F_k} \right)^{-1} \right\rfloor = \begin{cases} F_{n-2} & \text{if } n \geq 2 \text{ is even,} \\ F_{n-2} - 1 & \text{if } n \geq 1 \text{ is odd,} \end{cases}$$

where $\lfloor \cdot \rfloor$ denotes the floor function. This result has been investigated by several other mathematicians, see e.g. [**116**] and [**136**]. The above De Weger's result in [**70**] was extended in 2015 by Tengely [**243**] in which he obtained e.g.

$$\frac{1}{U_{10}} = \frac{1}{416020} = \sum_{k=1}^{\infty} \frac{U_{k-1}}{647^k},$$

where $U_0 = 0, U_1 = 1$ and $U_n = 4U_{n-1} + U_{n-2}, n \geq 2$. Moreover, there are many other nice results in the literature dealing with Diophantine equations related to base b representations and binary linear recurrence sequences. For instance, in 2016 Bravo and Luca [**35**] completely solved the equation

$$F_m + F_n = 2^a.$$

Recently, Chim and Ziegler [**55**] generalized their result in which they solved the equation

$$F_{n_1} + F_{n_2} = 2^{m_1} + 2^{m_2} + 2^{m_3}$$

in nonnegative integers $(n_1, n_2, m_1, m_2, m_3)$. Other related results dealing with Diophantine equations involving repdigits and linear recurrence sequences will be mentioned along with our results in the next section of this chapter, i.e. 2.2.

In this section, we present our new results in which we first extend the result of Tengely in [**243**] by determining the integral solutions (n, x) of the equation

$$\frac{1}{U_n(P_2, Q_2)} = \sum_{k=1}^{\infty} \frac{U_{k-1}(P_1, Q_1)}{x^k}, \tag{40}$$

for certain given pairs $(P_1, Q_1) \neq (P_2, Q_2)$. Here, we consider non-degenerate sequences with $1 \leq P \leq 3$ and $Q = \pm 1$ (for more details about the non-degenerate Lucas sequences, we refer to Subsection 1.3.1 of Chapter 1). Furthermore, we investigate the integral solutions (x, y) of the equation

$$\sum_{k=1}^{\infty} \frac{U_{k-1}(P, Q)}{x^k} = \sum_{k=1}^{\infty} \frac{R_{k-1}}{y^k}, \tag{41}$$

where the Lucas sequence of the first kind $\{U_n(P,Q)\}$ is non-degenerate with $1 \le P \le 3$ and $Q = \pm 1$, and the sequence $\{R_n\}$ is a ternary linear recurrence sequence represented by the Tribonacci sequence $\{T_n\}$ or Berstel's sequence $\{B_n\}$ that are defined by (17) or (18), respectively. We also provide general results related to the integral solutions (x, y) of the equations

$$\sum_{k=1}^{\infty} \frac{U_{k-1}(P_1, Q_1)}{x^k} = \sum_{k=1}^{\infty} \frac{U_{k-1}(P_2, Q_2)}{y^k}, \tag{42}$$

with arbitrary pairs $(P_1, Q_1) \ne (P_2, Q_2)$, and

$$\sum_{k=1}^{\infty} \frac{T_{k-1}(a_2, a_1, a_0)}{x^k} = \sum_{k=1}^{\infty} \frac{T_{k-1}(b_2, b_1, b_0)}{y^k}, \tag{43}$$

where the triples $(a_2, a_1, a_0) \ne (b_2, b_1, b_0)$ and T_n denotes the general term of the generalized Tribonacci sequence that is given by

$$T_0(p,q,r) = T_1(p,q,r) = 0, T_2(p,q,r) = 1 \quad \text{and}$$
$$T_n(p,q,r) = pT_{n-1}(p,q,r) + qT_{n-2}(p,q,r) + rT_{n-3}(p,q,r),$$

for $n \ge 3$. Then we apply these results to completely resolve some concrete equations.

Before presenting our new results, we mention some auxiliary results, which play the key roles in the proofs of our theorems.

2.1.1. Auxiliary results.

The following two results are due to Köhler [**135**].

LEMMA 2.1. *Let A, B, a_0, a_1 be arbitrary complex numbers. Define the sequence $\{a_n\}$ by the recursion $a_{n+1} = Aa_n + Ba_{n-1}$. Then the formula*

$$\sum_{k=1}^{\infty} \frac{a_{k-1}}{x^k} = \frac{a_0 x - Aa_0 + a_1}{x^2 - Ax - B}$$

holds for all complex x such that $|x|$ is larger than the absolute values of the zeros of $x^2 - Ax - B$.

LEMMA 2.2. *Let arbitrary complex numbers $A_0, A_1, \dots, A_m, a_0, a_1, \dots, a_m$ be given. Define the sequence $\{a_n\}$ by the recursion*

$$a_{n+1} = A_0 a_n + A_1 a_{n-1} + \cdots + A_m a_{n-m}.$$

Then for all complex z such that $|z|$ is larger than the absolute values of all zeros of $q(z) = z^{m+1} - A_0 z^m - A_1 z^{m-1} - \cdots - A_m$, the formula

$$\sum_{k=1}^{\infty} \frac{a_{k-1}}{z^k} = \frac{p(z)}{q(z)}$$

holds with $p(z) = a_0 z^m + b_1 z^{m-1} + \cdots + b_m$, where $b_k = a_k - \sum_{i=0}^{k-1} A_i a_{k-1-i}$ for $1 \le k \le m$.

2.1.2. New results.

Here, we present our new results related to the integral solutions of equations (40)-(43). The following two results are related to equations (40) and (41), respectively. Indeed, they are Theorems 1 and 2 in [**105**], respectively. For that, we define the set S as follows

$$S = \{u_1(n) = U_n(1,-1), u_2(n) = U_n(2,-1), u_3(n) = U_n(3,-1),\\ u_4(n) = U_n(3,1)\}.$$

THEOREM 2.3. *The equation*

$$\frac{1}{u_j(n)} = \sum_{k=1}^{\infty} \frac{u_i(k-1)}{x^k}, \tag{44}$$

has the following solutions with $1 \le i, j \le 4, i \ne j$

$$(i,j,n,x) \in \{(1,2,1,2),(1,2,3,3),(1,2,5,6),(1,3,1,2),(1,3,5,11),\\ (1,3,7,35),(1,4,1,2),(1,4,5,8),(2,1,3,3),(2,1,9,7),(3,1,4,4),\\ (3,1,14,21),(3,4,2,4),(3,4,7,21),(4,1,\{1,2\},3),(4,1,5,4),(4,\\ 1,10,9),(4,1,11,11),(4,2,1,3),(4,2,3,4),(4,2,5,7),(4,3,1,3),\\ (4,3,5,12),(4,3,7,36)\}.$$

THEOREM 2.4. *The complete list of solutions of the equation*

$$\sum_{k=1}^{\infty} \frac{u_j(k-1)}{x^k} = \sum_{k=1}^{\infty} \frac{R_{k-1}}{y^k}, \tag{45}$$

with $u_n \in S, R_n \in \{B_n, T_n\}$ *and positive integers* x, y *satisfying conditions of Lemmas 2.1 and 2.2 is as follows*

u_n	R_n	(x,y)	u_n	R_n	(x,y)
u_1	B_n	$\{(25,9)\}$	u_1	T_n	$\{(2,2)\}$
u_2	B_n	$\{\}$	u_2	T_n	$\{(t(t^2-2)+1, t^2-1) : t \ge 2, t \in \mathbb{N}\}$
u_3	B_n	$\{(6,3),(18,7)\}$	u_3	T_n	$\{\}$
u_4	B_n	$\{(26,9)\}$	u_4	T_n	$\{(3,2)\}$

Using elementary number theory, we prove the following results, which are related to equations (42) and (43), respectively. These results are Theorem 1 and Theorem 2 in [**106**], respectively. For the sake of simplicity in presenting these results, we define the following. For a given polynomial $f(x)$ over integers, let $m(f) = \max\{|x| : f(x) = 0\}$.

THEOREM 2.5. *Let* P_1, Q_1, P_2, Q_2 *be non-zero integers such that* $(P_1, Q_1) \ne (P_2, Q_2)$. *If* $(P_2^2 - P_1^2) + 4(Q_1 - Q_2) = d_1 d_2 \ne 0$ *and* $d_1 - d_2 \equiv -2P_1 \pmod 4), d_1 + d_2 \equiv$

$-2P_2 \pmod 4$, *then the positive integral solutions* x, y *of*

$$\sum_{k=1}^{\infty} \frac{U_{k-1}(P_1, Q_1)}{x^k} = \sum_{k=1}^{\infty} \frac{U_{k-1}(P_2, Q_2)}{y^k}$$

satisfy

$$x = \frac{d_1 - d_2 + 2P_1}{4} > m(x^2 - P_1 x + Q_1), \quad y = \frac{d_1 + d_2 + 2P_2}{4} > m(x^2 - P_2 x + Q_2).$$

If $(P_2^2 - P_1^2) + 4(Q_1 - Q_2) = 0$ *and* $P_1 \equiv P_2 \pmod 2$, *then the positive integral solutions* x, y *of*

$$\sum_{k=1}^{\infty} \frac{U_{k-1}(P_1, Q_1)}{x^k} = \sum_{k=1}^{\infty} \frac{U_{k-1}(P_2, Q_2)}{y^k}$$

satisfy

$$x > m(x^2 - P_1 x + Q_1), \quad y = \pm x + \frac{P_2 \mp P_1}{2} > m(x^2 - P_2 x + Q_2),$$

where $Q_2 = Q_1 + \frac{P_2^2 - P_1^2}{4}$.

THEOREM 2.6. *If* (x, y) *is an integral solution of the equation*

$$\sum_{k=1}^{\infty} \frac{T_{k-1}(a_2, a_1, a_0)}{x^k} = \sum_{k=1}^{\infty} \frac{T_{k-1}(b_2, b_1, b_0)}{y^k},$$

for given $(a_2, a_1, a_0) \neq (b_2, b_1, b_0)$, *then either*

$$9\left(a_2^2 - b_2^2 + 3\,a_1 - 3\,b_1\right)y + 2\,a_2^3 - 3\,a_2^2 b_2 + b_2^3 + 9\,a_1 a_2 - 9\,a_1 b_2 + 27\,a_0 - 27\,b_0 = 0$$

or in case of $|y| > B$ *we have*

$$|3x - 3y - a_2 + b_2| < C,$$

where B, C *are constants depending only on* $a_i, b_i, i = 0, 1, 2$.

2.1.3. Proofs of the results.

PROOF OF THEOREM 2.3. Consider equation (40) (in particular, (44)), by Lemma 2.1 we obtain that

$$\sum_{k=1}^{\infty} \frac{U_{k-1}(P_1, Q_1)}{x^k} = \frac{1}{x^2 - P_1 x + Q_1}.$$

Hence, we have that $U_n(P_2, Q_2) = x^2 - P_1 x + Q_1$. Combining the latter equation with the identity relationship between the terms of Lucas sequences of the first and second kind at the parameters P_2 and Q_2, which is defined in (21) (i.e. $V_n^2(P_2, Q_2) = DU_n^2(P_2, Q_2) + 4Q_2^n$, with $D = P_2^2 - 4Q_2$) we get

$$V_n(P_2, Q_2)^2 = (P_2^2 - 4Q_2)(x^2 - P_1 x + Q_1)^2 + 4Q_2^n.$$

The so-called two-cover descent by Bruin and Stoll [**40**] can be used to prove that a given hyperelliptic curve has no rational points. It is implemented in Magma [**33**], the procedure is called `TwoCoverDescent`. If it fails and we do not find any rational points on the curve, then we apply the argument by Alekseyev and Tengely [**4**], that reduces the problem to Thue equations. If we have a rational point on the curve, then using a method by Tzanakis in [**249**] the integral points can be determined. This algorithm is implemented in Magma as `IntegralQuarticPoints`. In this way we collect the possible values of x.

(P_1,Q_1,P_2,Q_2)	x	(P_1,Q_1,P_2,Q_2)	x
$(1,-1,2,-1)$	$2,3,6$	$(3,-1,1,-1)$	$4,21$
$(1,-1,3,-1)$	$2,11,35$	$(3,-1,2,-1)$	$-$
$(1,-1,3,1)$	$2,8$	$(3,-1,3,1)$	$4,21$
$(2,-1,1,-1)$	$3,7$	$(3,1,1,-1)$	$3,4,9,11$
$(2,-1,3,-1)$	$-$	$(3,1,2,-1)$	$3,4,7$
$(2,-1,3,1)$	$-$	$(3,1,3,-1)$	$3,12,36$

It remains to compute the set of possible values of n. We provide details of the computation in case of $(P_1,Q_1,P_2,Q_2)=(3,-1,1,-1)$, and following these steps all other equations can be handled. In case of $(P_1,Q_1,P_2,Q_2)=(3,-1,1,-1)$ we have that $x\in\{4,21\}$. If $x=4$, then we define a matrix T as follows

$$T=\begin{pmatrix}3/4 & 1/4\\ 1/4 & 0\end{pmatrix}.$$

We have that

$$\frac{1}{4}\left(T^0+T^1+T^2+\cdots+T^{N-1}\right)\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix} * \\ \sum_{k=1}^N \frac{U_{k-1}(3,-1)}{4^k}\end{pmatrix}.$$

It follows that

$$\sum_{k=1}^N \frac{U_{k-1}(3,-1)}{4^k}=-\frac{2^{-3N-1}}{39}\left(\left(\sqrt{13}+3\right)^N\left(5\sqrt{13}+13\right)+\left(13-5\sqrt{13}\right)\left(-\sqrt{13}+3\right)^N-13\cdot 2^{3N+1}\right).$$

Hence, we have that

$$\lim_{N\to\infty}\sum_{k=1}^N \frac{U_{k-1}(3,-1)}{4^k}=\frac{1}{3}=\frac{1}{U_4(1,-1)}.$$

In this case, we obtain that $n=4$. If $x=21$, then

$$T=\begin{pmatrix}3/21 & 1/21\\ 1/21 & 0\end{pmatrix}.$$

In a similar way than in case of $x=4$, we get that

$$\sum_{k=1}^N \frac{U_{k-1}(3,-1)}{21^k}=\frac{\left(7^N3^N2^{N+1}-(\sqrt{13}+3)^N(3\sqrt{13}+1)+(3\sqrt{13}-1)(-\sqrt{13}+3)^N\right)2^{-N-1}}{377\cdot 7^N3^N}.$$

Therefore,

$$\lim_{N\to\infty}\sum_{k=1}^N \frac{U_{k-1}(3,-1)}{21^k}=\frac{1}{377}=\frac{1}{U_{14}(1,-1)}.$$

The only solution in this case is given by $n=14$. □

PROOF OF THEOREM 2.4. We provide a general argument that works for other sequences as well. Let

$$a_0 = 0, a_1 = 1 \quad \text{and} \quad a_{n+1} = Aa_n + Ba_{n-1}$$

and

$$b_0 = b_1 = 0, b_2 = 1 \quad \text{and} \quad b_{n+1} = Cb_n + Db_{n-1} + Eb_{n-2}.$$

Equation (45) yields that

$$Y^2 = X^3 - 4CX^2 - 16DX + 16A^2 + 64B - 64E,$$

where $Y = 8x - 4A$ and $X = 4y$. If the cubic polynomial in X is square-free, then we have an elliptic equation and the integral points can be determined using the so-called elliptic logarithm method developed by Stroeker and Tzanakis [**233**] and independently by Gebel, Pethő and Zimmer [**91**]. There exists a number of software implementations for determining integral points on elliptic curves based on this technique, here we used SageMath [**231**]. Let us consider the case with $u_1(n), T_n$. We obtain the elliptic curve

$$Y^2 = X^3 - 4X^2 - 16X + 16.$$

Using the SageMath function `integral_points()` we get

$$[(-3 : 1 : 1), (0 : 4 : 1), (8 : 12 : 1)].$$

From these points we have that $(x, y) \in \{(2, 2)\}$. As a second example, consider the case with u_3, B_n. The elliptic curve is given by

$$Y^2 = X^3 - 8X^2 + 64X - 48.$$

The list of integral points is

$$[(1 : 3 : 1), (4 : 12 : 1), (12 : 36 : 1), (28 : 132 : 1)].$$

Thus, we get that $(x, y) \in \{(6, 3), (18, 7)\}$. Finally, let us deal with the special case with u_2, T_n. The cubic polynomial is not square-free, it is $(X+4)(X-4)^2$. Therefore, we have that

$$X + 4 = 4y + 4 = u^2.$$

Hence, $y = t^2 - 1$ for some integer $t \geq 2$. It follows that $x = t(t^2 - 2) + 1$. So we obtain infinitely many identities of the form

$$\sum_{k=1}^{\infty} \frac{u_2(k-1)}{(t(t^2-2)+1)^k} = \sum_{k=1}^{\infty} \frac{T_{k-1}}{(t^2-1)^k}.$$

□

PROOF OF THEOREM 2.5. By applying Lemma 2.1 to equation (42), we get that

$$\frac{1}{x^2 - P_1x + Q_1} = \frac{1}{y^2 - P_2y + Q_2}.$$

By algebraic manipulations, we obtain the equation

$$(2y + 2x - P_1 - P_2)(2y - 2x + P_1 - P_2) = P_2^2 - P_1^2 + 4(Q_1 - Q_2). \qquad (46)$$

If $P_2^2 - P_1^2 + 4(Q_1 - Q_2) \neq 0$, then for all $d_1 | P_2^2 - P_1^2 + 4(Q_1 - Q_2)$ we consider the following system of equations

$$\begin{aligned} 2y + 2x - P_1 - P_2 &= d_1, \\ 2y - 2x + P_1 - P_2 &= d_2 = \frac{P_2^2 - P_1^2 + 4(Q_1 - Q_2)}{d_1}. \end{aligned}$$

We obtain integral solutions if

$$d_1 - d_2 \equiv -2P_1 \pmod 4 \text{ and } d_1 + d_2 \equiv -2P_2 \pmod 4.$$

In this case, the solutions are given by

$$x = \frac{d_1 - d_2 + 2P_1}{4} \text{ and } y = \frac{d_1 + d_2 + 2P_2}{4}.$$

If $P_2^2 - P_1^2 + 4(Q_1 - Q_2) = 0$, then

$$Q_2 = Q_1 + \frac{P_2^2 - P_1^2}{4},$$

and Q_2 is an integer if $P_1 \equiv P_2 \pmod 2$. There are two possible cases, either

$$2y + 2x - P_1 - P_2 = 0 \quad \text{or} \quad 2y - 2x + P_1 - P_2 = 0.$$

In the former case we have $y = -x + \frac{P_1+P_2}{2}$ and in the latter one we get $y = x + \frac{P_2 - P_1}{2}$. □

Proof of Theorem 2.6. Using Köhler's result given in Lemma 2.2, equation (43) yields that

$$\frac{1}{x^3 - a_2x^2 - a_1x - a_0} = \frac{1}{y^3 - b_2y^2 - b_1y - b_0}.$$

Hence,

$$H(x,y) = x^3 - a_2x^2 - a_1x - a_0 - y^3 + b_2y^2 + b_1y + b_0 = 0.$$

This equation satisfies Runge's condition [**200**]. Therefore, in case when $H(x,y)$ is irreducible over the rationals, there exist only finitely many integral solutions (x,y). We obtain that

$$0 = 27H(x,y) = (3x - 3y - a_2 + b_2)G(x,y) - I(y),$$

where

$$G(x,y) = 9x^2 + 9xy + 9y^2 - 3(2a_2 + b_2)x - 3(a_2 + 2b_2)y - 2a_2^2 + a_2b_2 + b_2^2 - 9a_1$$

and

$$I(y) = 9\left(a_2^2 - b_2^2 + 3a_1 - 3b_1\right)y + 2a_2^3 - 3a_2^2b_2 + b_2^3 + 9a_1a_2 - 9a_1b_2 + 27a_0 - 27b_0.$$

Here, we may have that $I(y)$ is identically equal to 0, then $H(x,y)$ is reducible over the rationals. In this case, solutions can be obtained from

$$3x - 3y - a_2 + b_2 = 0 \quad \text{or} \quad G(x,y) = 0.$$

Now, assume that $I(y) \neq 0$. We obtain that

$$\begin{aligned} 3x - 3y - a_2 + b_2 &= \frac{I(y)}{G(x,y)} \\ &= \frac{4I(y)}{27\,y^2 - 18\,b_2 y - 12\,a_2^2 + 3\,b_2^2 - 36\,a_1 + (6x + 3y - (2a_2 + b_2))^2}. \end{aligned}$$

Here, one can determine a bound B for $|y|$ such that if $|y| > B$, then

$$27\,y^2 - 18\,b_2 y - 12\,a_2^2 + 3\,b_2^2 - 36\,a_1 + (6x + 3y - (2a_2 + b_2))^2 > 26y^2.$$

Thus,

$$|3x - 3y - a_2 + b_2| < \frac{4I(y)}{26y^2}.$$

Since $I(y)$ is linear, we get that

$$\frac{4|I(y)|}{26y^2} < C$$

for some positive constant C and the statement follows. □

Next, we present some applications related to Theorems 2.5 and 2.6. Indeed, these applications also appear in [**106**].

2.1.4. Applications.

EXAMPLE 2.1. *As an application of Theorem 2.5, consider the following example. Let* $(P_1, Q_1) = (1, -1)$ *and* $(P_2, Q_2) = (18, 1)$. *We have that*

$$(P_2^2 - P_1^2) + 4(Q_1 - Q_2) = (18^2 - 1) + 4(-1 - 1) = 315.$$

We obtain a system of equations given by

$$\begin{aligned} 2y + 2x - 19 &= d_1, \\ 2y - 2x - 17 &= \frac{315}{d_1}, \end{aligned}$$

where

$$d_1 \in \{\pm 1, \pm 3, \pm 5, \pm 7, \pm 9, \pm 15, \pm 21, \pm 35, \pm 45, \pm 63, \pm 105, \pm 315\}.$$

The solutions are as follows

$$\begin{aligned} (x,y) \in \; & \{(79,-70), (26,-18), (15,-8), (10,-4), (7,-2), (2,0), (-1,0), \\ & (-6,-2), (-9,-4), (-14,-8), (-25,-18), (-78,-70), (-78,88), \\ & (-25,36), (-14,26), (-9,22), (-6,20), (-1,18), (2,18), (7,20), \\ & (10,22), (15,26), (26,36), (79,88)\}. \end{aligned}$$

Here, we have $m(x^2-x-1)\approx 1.618$*, so* $x\geq 2$ *and* $m(x^2-18x+1)\approx 17.944$*. Hence,* $y\geq 18$*. Thus, the solutions are as follows*

$$\begin{aligned}
\sum_{k=1}^{\infty}\frac{U_{k-1}(1,-1)}{2^k}=\sum_{k=1}^{\infty}\frac{U_{k-1}(18,1)}{18^k} &= 1,\\
\sum_{k=1}^{\infty}\frac{U_{k-1}(1,-1)}{7^k}=\sum_{k=1}^{\infty}\frac{U_{k-1}(18,1)}{20^k} &= \frac{1}{41},\\
\sum_{k=1}^{\infty}\frac{U_{k-1}(1,-1)}{10^k}=\sum_{k=1}^{\infty}\frac{U_{k-1}(18,1)}{22^k} &= \frac{1}{89},\\
\sum_{k=1}^{\infty}\frac{U_{k-1}(1,-1)}{15^k}=\sum_{k=1}^{\infty}\frac{U_{k-1}(18,1)}{26^k} &= \frac{1}{209},\\
\sum_{k=1}^{\infty}\frac{U_{k-1}(1,-1)}{26^k}=\sum_{k=1}^{\infty}\frac{U_{k-1}(18,1)}{36^k} &= \frac{1}{649},\\
\sum_{k=1}^{\infty}\frac{U_{k-1}(1,-1)}{79^k}=\sum_{k=1}^{\infty}\frac{U_{k-1}(18,1)}{88^k} &= \frac{1}{6161}.
\end{aligned}$$

EXAMPLE 2.2. *As a next example, consider the case with* $(P_1,Q_1)=(1,-1)$ *and* $(P_2,Q_2)=(2t+1,t^2+t-1)$ *for some* $t\in\mathbb{Z}$*. We get that*

$$\sum_{k=1}^{\infty}\frac{U_{k-1}(1,-1)}{x^k}=\sum_{k=1}^{\infty}\frac{U_{k-1}(2t+1,t^2+t-1)}{(x+t)^k}=\frac{1}{x^2-x-1}$$

for $x\geq 2$.

EXAMPLE 2.3. *Consider the positive integral solutions* x,y *of the equation*

$$\sum_{k=1}^{\infty}\frac{T_{k-1}(-1,7,3)}{x^k}=\sum_{k=1}^{\infty}\frac{T_{k-1}(5,-5,-3)}{y^k}.$$

Lemma 2.2 implies that

$$\frac{1}{x^3+x^2-7x-3}=\frac{1}{y^3-5y^2+5y+3}.$$

Therefore, we get that

$$x^3+x^2-7x-3=y^3-5y^2+5y+3.$$

Following the proof of Theorem 2.6, we have that

$$H(x,y)=x^3+x^2-7x-y^3+5y^2-5y-6=0.$$

We determine $G(x,y)$ *and* $I(y)$*, these are given by*

$$G(x,y)=9x^2+9xy+9y^2-9x-27y-45,\quad I(y)=108y-108.$$

If $I(y)=0$*, then* $y=1$ *and it follows that*

$$x^3+x^2-7x-3=4,$$

that is $x = -1$. *Hence, we do not get positive integral solutions in this case. Assume that* $I(y) \neq 0$. *We get that*

$$3x - 3y + 6 = \frac{4(108y - 108)}{9x^2 + 9xy + 9y^2 - 9x - 27y - 45}.$$

It can be written as

$$3x - 3y + 6 = \frac{4(108y - 108)}{27y^2 - 90y - 189 + (6x + 3y - 3)^2}.$$

We have that

$$26y^2 < 27y^2 - 90y - 189 + (6x + 3y - 3)^2$$

for positive integers if $y \geq 93$. *It follows that*

$$|3x - 3y + 6| < 1 \quad \textit{if} \quad y \geq 93.$$

That is $3x - 3y + 6 = 0$, *so we obtain that* $I(y) = 0$, *a contradiction. It remains to deal with the values of y for which*

$$3 = m(x^3 - 5x^2 + 5x + 3) \leq y \leq 93.$$

Using SageMath [**231**], *we obtain that the only integral solutions in this range are given by* $x = -3, y = 3$ *and* $x = -2, y = 4$, *so we do not get positive integral solutions.*

EXAMPLE 2.4. *As a second application of Theorem 2.6, let us consider the equation*

$$\sum_{k=1}^{\infty} \frac{T_{k-1}(-4,-5,-6)}{x^k} = \sum_{k=1}^{\infty} \frac{T_{k-1}(1,8,18)}{y^k}.$$

Here, we obtain that

$$\begin{aligned} H(x,y) &= x^3 - y^3 + 4x^2 + y^2 + 5x + 8y + 24, \\ G(x,y) &= 9x^2 + 9xy + 9y^2 + 21x + 6y + 10, \\ I(y) &= -216y - 598. \end{aligned}$$

The equation $I(y) = 0$ *does not have integral solutions. We obtain that*

$$4G(x,y) > 26y^2 \quad \textit{if} \quad y > 18$$

and

$$|3x - 3y + 5| < 1 \quad \textit{if} \quad y > 30.$$

Hence, we have that if $y > 30$, *then* $3x - 3y + 5 = 0$. *Therefore,* $I(y) = 0$, *a contradiction. It remains to deal with the cases* $y \in [5, 6, \ldots, 30]$. *It follows that the only positive solution is given by* $(x, y) = (9, 11)$, *that is we have*

$$\sum_{k=1}^{\infty} \frac{T_{k-1}(-4,-5,-6)}{9^k} = \sum_{k=1}^{\infty} \frac{T_{k-1}(1,8,18)}{11^k} = \frac{1}{1104}.$$

EXAMPLE 2.5. *Finally, let us describe an example with identically zero* $I(y)$*, in which case we obtain infinitely many solutions. Let* $(a_2, a_1, a_0) = (1, 6, 5)$ *and* $(b_2, b_1, b_0) = (4, 1, 1)$*. It follows that*

$$\begin{aligned} H(x,y) &= x^3 - y^3 - x^2 + 4y^2 - 6x + y - 4, \\ G(x,y) &= 9x^2 + 9xy + 9y^2 - 18x - 27y - 36, \\ I(y) &= 0. \end{aligned}$$

We obtain that either

$$3x - 3y + 3 = 0 \quad \text{or} \quad G(x,y) = 0.$$

In the former case $y = x + 1$ *and*

$$\sum_{k=1}^{\infty} \frac{T_{k-1}(1,6,5)}{x^k} = \sum_{k=1}^{\infty} \frac{T_{k-1}(4,1,1)}{(x+1)^k} = \frac{1}{x^3 - x^2 - 6x - 5}, \qquad x \geq 4.$$

In the latter case we have that

$$0 = 12G(x,y) = 3(6x + 3y - 6)^2 + (9y - 12)^2 - 684.$$

We do not get new integral solutions since the equation

$$(9y - 12)^2 + 3(6x + 3y - 6)^2 = 684$$

has no solutions in $\mathbb{Z}$*.*

2.2. Diophantine equations related to repdigits with linear recurrence sequences

A natural number N is called a base g-repdigit for $g \geq 2$ if all of its base g-digits are equal; that is, if

$$N = b \cdot \left(\frac{g^m - 1}{g - 1}\right) \quad \text{for some} \quad m \geq 1 \quad \text{and} \quad b \in \{1, \ldots, g-1\}.$$

Diophantine equations involving repdigits and recurrence sequences have been studied in several papers. The problem of finding all perfect powers among repdigits was presented in 1956 by Obláth [**180**] and then solved by Bugeaud and Mignotte [**46**] in 1999. Thereafter, many authors have started to investigate the solutions of such Diophantine equations. For instance, Luca [**151**] used an elementary way to show that the largest number whose decimal expansion has only one distinct digit in the sequence of Fibonacci numbers or Lucas numbers is $F_{10} = 55$ or $L_5 = 11$, respectively (the terms of the Fibonacci sequence $\{F_n\}$ and Lucas sequence $\{L_n\}$ are given by (23) and (25), respectively). In addition, Díaz-Alvarado and Luca [**71**] found all Fibonacci numbers that are sums of two repdigits. Furthermore, a similar problem was investigated in case of Lucas numbers by Adegbindin, Luca and Togbé [**2**]. For other related results, we refer to [**36**], [**62**], [**82**], [**124**], [**164**] and the references given there. Luca's result in [**151**] was generalized by Marques and Togbé [**163**] in which they determined all the solutions of the Diophantine equations

$$F_n = B \cdot \Big(\frac{10^{lm} - 1}{10^l - 1} \Big) \quad \text{and} \quad L_n = B \cdot \Big(\frac{10^{lm} - 1}{10^l - 1} \Big) \tag{47}$$

in positive integers m, n and l, with $m > 1, 1 \le l \le 10$ and $1 \le B \le 10^l - 1$, which are $(m, n, l) = (2, 10, 1)$ and $(m, n, l) = (2, 5, 1)$ in the Fibonacci and Lucas cases, respectively. It is clear that these equations have solutions only with $l = 1$. In general, if $(G_n)_{n \ge 1}$ is an integer linear recurrence sequence, they gave a finiteness result for the equation

$$G_n = B \cdot \Big(\frac{g^{lm} - 1}{g^l - 1} \Big), \tag{48}$$

where n, m, g, l and B are positive integers such that $m > 1, g > 1, 1 \le B \le g^l - 1$. In fact, they proved their results using heavy computations followed by a result due to Matveev [**166**] on the lower bound on linear forms of logarithms of algebraic numbers to obtain bounds for n and m. As these bounds could be very high, they used a result due to Dujella and Pethő [**76**] on the Baker-Davenport reduction to reduce these bounds. With respect to these results, the following natural questions arise:

- Is there another approach that is easier to apply to such concrete equations?
- Do the equations in (47) have solutions in any base g other than 10, say $g \ge 2$, in case of $l = 1$?

In this section, we firstly use a different and direct approach to obtain a general finiteness result for the Diophantine equation (48) in which the sequence $\{G_n\}$ is represented by the non-degenerate Lucas sequences of the first and second kind (i.e. $\{U_n(P, Q)\}$ and $\{V_n(P, Q)\}$, which are defined by (19) and (20), respectively) with $Q \in \{-1, 1\}$ and l is an even positive integer. Again, we refer to Subsection 1.3.1 of Chapter 1 for more details about the non-degenerate Lucas sequences. Our argument is based on combining equation (48) with the identity relationship between Lucas sequences of the first and second kind (21) (that is $V_n^2 = DU_n^2 + 4Q^n$, with $D = P^2 - 4Q$) to produce biquadratic elliptic curves of the form

$$y^2 = ax^4 + bx^2 + c, \tag{49}$$

with integer coefficients a, b, c and discriminant $\Delta = 16ac(b^2 - 4ac)^2 \ne 0$. The integral points of a biquadratic elliptic curve can be determined using an algorithm implemented in Magma [**33**] as `SIntegralLjunggrenPoints()` (based on results obtained by Tzanakis [**249**]) or an algorithm described by Alekseyev and Tengely [**4**] in which they gave an algorithmic reduction of the search for integral points on such a curve by solving a finite number of Thue equations. It is clear that such a biquadratic elliptic curve in the form (49) can be further written as an elliptic curve of the form

$$Y^2 = X^3 + bX^2 + acX, \tag{50}$$

where $X = ax^2, Y = axy$ and its discriminant is $a^2c^2(b^2 - 4ac) \ne 0$. As mentioned earlier in the proof of Theorem 2.4 in Section 2.1, for determining all integral points on a given elliptic curve one can follow the so-called elliptic logarithm method developed by Stroeker and Tzanakis [**233**] and independently by Gebel, Pethő and Zimmer [**91**]. Thus, for determining the integral points on such elliptic curves based on this technique, we use the SageMath [**231**] function `integral_points()`. The finiteness of the number of the integral points on the curves (49) or (50) is guaranteed by Baker's result [**11**] presented in Theorem 1.16 and its best improvement concerning the solutions of elliptic equations over $\mathbb{Q}$, that is due to Hajdu and Herendi [**103**]. For simplicity, let us recall this result in the following theorem and call it by Baker's Theorem:

Baker's Theorem: If the polynomial on the right of the Diophantine equation

$$y^2 = a_0x^n + a_1x^{n-1} + \ldots + a_n,$$

where $n \geq 3$ and $a_0 \neq 0, a_1, \ldots, a_n \in \mathbb{Z}$, possesses at least three simple zeros, then all of its solutions in integers x, y satisfy

$$\max(|x|, |y|) < \exp\exp\exp\{(n^{10n}H)^{n^2}\},$$

where $H = \max_{0\leq i\leq n} |a_i|$.

For more details about Baker's results and their improvements, we refer to Subsection 1.2.4 of Chapter 1.

REMARK 2.7. *Since a finiteness result for equation* (48) *in case of $G_n = U_n$ or $G_n = V_n$ can be obtained in a similar way, we only present and prove this result in detail in the case of $G_n = U_n$ and omit the proof of the remaining case.*

As applications of our result, we apply our method on the sequence of Fibonacci numbers $\{F_n\}$ and the sequence of Pell numbers $\{P_n\}$ (which is defined by (24)) that satisfy equation (48). Furthermore, with the first application we also generalize the result of Marques and Togbé in [**163**] in case of Fibonacci numbers by determining all the solutions (n, m, g, B, l) of the equation $F_n = B \cdot \left(\frac{g^{lm}-1}{g^l-1}\right)$ in case of $2 \leq g \leq 9$ and $l = 1$. Note that the case of Lucas numbers can be generalized similarly, therefore we omit the details of this case. More precisely, we use our approach in case where we have l is even, otherwise we follow the technique of Marques and Togbé in [**163**] of using the result of Matveev on linear forms in three logarithms and the result of Dujella and Pethő on the method of Baker-Davenport reduction.

For the sake of simplicity, we next recall some useful results that will be used in the proof of Theorem 2.11 of our main results (particularly, in case of $l = 1$).

2.2.1. Auxiliary results.

From (31) and (35), we respectively recall the Binet's Fibonacci numbers formula and the bounds for the n^{th} Fibonacci number as follows. The Binet's Fibonacci numbers formula is defined as

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad \text{where} \quad (\alpha, \beta) = \left(\frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}\right) \tag{51}$$

for all $n \geq 0$, where α is called the golden ratio and $\beta = \frac{-1}{\alpha}$. Moreover, it is also known that

$$\alpha^{n-2} \leq F_n \leq \alpha^{n-1} \quad \text{holds for all} \quad n \geq 1. \tag{52}$$

Also, we recall the logarithmic height of an s-degree algebraic number α that is defined as

$$h(\alpha) = \frac{1}{s}(\log|a| + \sum_{j=1}^{s} \log\max\{1, \left|\alpha^{(j)}\right|\}), \tag{53}$$

where a is the leading coefficient of the minimal polynomial of α (over $\mathbb{Z}$), $(\alpha^{(j)})_{1\leq j\leq s}$ are the conjugates of the algebraic number α, and the absolute value of a complex number $z = x + iy$ is determined by $|z| = \sqrt{x^2 + y^2}$.

In order to obtain some bounds for n and m, we need to use a lower bound for a linear form logarithms á la Baker, which was given by the following lemma due to Matveev [**166**] (also see Lemma 2 in [**163**]).

LEMMA 2.8. *Define*

$$\Lambda = a_1 \log\alpha_1 + a_2 \log\alpha_2 + a_3 \log\alpha_3, \tag{54}$$

where a_1, a_2 and a_3 are nonzero integers and α_1, α_2 and α_3 are nonzero algebraic numbers. Let d be the degree of the number field $\mathbb{Q}(\alpha_1, \alpha_2, \alpha_3)$ over $\mathbb{Q}$ and $\chi = [\mathbb{R}(\alpha_1, \alpha_2, \alpha_3) : \mathbb{R}]$. If $\Lambda \neq 0$, then

$$\log|\Lambda| \geq -C_1 d^2 A_1 A_2 A_3 \log(1.5edB' \log(ed)),$$

where A_1, A_2 and A_3 are real numbers satisfying the condition

$$A_j \geq \max\{dh(\alpha_j), |\log(\alpha_j)|, 0.16\}, \text{ for all } j \in \{1, 2, 3\}, \tag{55}$$

$$B' \geq \max\{1, \max\{|a_j| A_j/A_1 : 1 \leq j \leq 3\}\}, \tag{56}$$

and

$$C_1 = \frac{5.16^5}{6\chi}.e^3(7 + 2\chi)(20.2 + \log(3^{5.5}d^2 \log(ed))). \tag{57}$$

After finding upper bounds for n and m, which could be very large, the next step is we have to reduce them. For that, we use the following lemma, which is a variant of the Baker-Davenport lemma, due to the result of Dujella and Pethő (see Lemma 5 in [**76**]).

LEMMA 2.9. *Suppose that M is a positive integer. Let p/q be the convergent of the continued fraction expansion of κ such that $q > 6M$ and let $\epsilon = \|\mu q\| - M\|\kappa q\|$, where $\|\cdot\|$ denotes the distance from the nearest integer. If $\epsilon > 0$, then there is no solution of the inequality*

$$0 < m\kappa - n + \mu < AK^{-m}$$

in $n, m \in \mathbb{Z}$ *in the range*

$$\frac{\log(Aq/\epsilon)}{\log(K)} \le m \le M.$$

2.2.2. New results.

Here, we present two main theorems. The first one is our main theoretical result related to the finiteness result of equation (48) in case of l is even, and in the second theorem we present our computational results regarding to applications of our method on the sequences of Fibonacci and Pell numbers and a generalization of the Fibonacci case in the result of Marques and Togbé in [**163**]. We also remark that in Theorem 2.10; $g \ge 2$, l is even and $1 \le B \le g^l - 1$ are assumed. But in practice, determining the solutions of any equation of the form (48) (for a particular sequence and any even number l) is achieved similarly for every $g \in \{2, 3, \ldots\}$ and $B \in \{1, 2, \ldots, g^l - 1\}$. Therefore, to make the presentation simpler, in Theorem 2.11 we assume that $2 \le g \le 9, l \in \{1, 2, 4\}$ and $1 \le B \le \min\{10, g^l - 1\}$ in case of $G_n = F_n$ and $2 \le g \le 9, l = 2$ and $1 \le B \le \min\{5, g^l - 1\}$ in case of $G_n = P_n$. These results are obtained in [**108**].

THEOREM 2.10. *Let* P *and* Q *be nonzero relatively prime integers with* $Q \in \{-1, 1\}$ *and* t *be a positive integer. If* $G_n = U_n(P, Q)$ *is non-degenerate and* $l = 2t$*, then the Diophantine equation* (48) *has finitely many solutions of the form* (n, m, g, B, l)*, which can be effectively determined.*

THEOREM 2.11. *If* $G_n = F_n$*, then the Diophantine equation* (48) *has the following solutions with* $2 \le g \le 9, l \in \{1, 2, 4\}$ *and* $1 \le B \le \min\{10, g^l - 1\}$.

$$\begin{aligned}(n, m, g, B, l) \in \{&(4,2,2,1,1), (5,2,4,1,1), (6,2,3,2,1), (6,2,7,1,1),\\ &(7,3,3,1,1), (8,2,6,3,1), (8,3,4,1,1), (5,2,2,1,2), (8,3,2,1,2),\\ &(9,2,4,2,2), (9,2,2,2,4)\}.\end{aligned}$$

Furthermore, suppose that $2 \le g \le 9, l = 2, 1 \le B \le \min\{5, g^l - 1\}$ *and* $G_n = P_n$*, then equation* (48) *has no more solutions other than* $(n, m, g, B, l) = (3, 2, 2, 1, 2)$.

2.2.3. Proofs of the results.

PROOF OF THEOREM 2.10. Since $G_n = U_n(P, Q) = U_n$ with $Q \in \{-1, 1\}$ and $l = 2t$ for an integer $t \ge 1$, we combine equation (48) with identity (21) to obtain

$$(g^{2t} - 1)^2 V_n^2 = DB^2(g^{2tm} - 1)^2 + 4(g^{2t} - 1)^2 Q^n,$$

which can be further written as biquadratic curves of the form

$$y^2 = DB^2(x^4 - 2x^2 + 1) + 4G^2 Q^n, \tag{58}$$

where $D = P^2 - 4Q, G = (g^{2t} - 1), 1 \le B \le (g^{2t} - 1), x = g^{tm}$ and $y = GV_n$ such that $m, g \ge 2$. Next, we show that the given curves have nonzero discriminants in order to prove they present elliptic curves. Since $Q \in \{-1, 1\}$, we split the prove into two cases

□ `Case` 1. If $Q = 1$, then equation (58) becomes

$$y^2 = D_1 B^2(x^4 - 2x^2 + 1) + 4G^2, \tag{59}$$

where $D_1 = (P^2 - 4)$, whose discriminant is

$$\Delta_1 = 4096 D_1^3 G^4 B^6 (D_1 B^2 + 4G^2).$$

In addition to P is nonzero, we consider only non-degenerate Lucas sequences, i.e.

$$(P, Q) \notin \{(-2, 1), (-1, 1), (1, 1), (2, 1)\}.$$

Hence, $D_1 > 0$, which implies that $(D_1 B^2 + 4G^2) > 0$ as $B > 0$ and $G > 0$. Therefore, it is clear that $\Delta_1 \neq 0$ (indeed, $\Delta_1 > 0$).

□ `Case` 2. Similarly, if $Q = -1$, we obtain the curves

$$y^2 = D_2 B^2(x^4 - 2x^2 + 1) + 4G^2 \tag{60}$$

and

$$y^2 = D_2 B^2(x^4 - 2x^2 + 1) - 4G^2, \tag{61}$$

where $D_2 = (P^2 + 4)$, and their discriminants are

$$\Delta_2 = 4096 D_2^3 G^4 B^6 (D_2 B^2 + 4G^2)$$

and

$$\Delta_3 = 4096 D_2^3 G^4 B^6 (D_2 B^2 - 4G^2),$$

respectively. Again, it is obvious that $\Delta_2 \neq 0$ as $D_2 > 0, B > 0$ and $G > 0$. For a contradiction, we assume that $\Delta_3 = 0$, which is true if and only if

$$D_2 B^2 - 4G^2 = 0.$$

The latter equation is true if and only if D_2 is a square number; that is, if there exists a nonzero integer T such that

$$T^2 - P^2 = 4,$$

which has no more rational integer solutions other than $(T, P) \in \{(-2, 0), (2, 0)\}$, which contradicts that $P \neq 0$. Thus, $\Delta_3 \neq 0$.

Therefore, we conclude that the biquadratic curves (58) represent elliptic curves. Moreover, as mentioned earlier that the biquadratic curves (58) can be written in the form (50); that is,

$$Y^2 = X^3 - 2DB^2X^2 + DB^2(DB^2 + 4G^2Q^n)X, \tag{62}$$

where $X = DB^2x^2$ and $Y = DB^2xy$. In a similar way, one can easily show that the latter curves have nonzero discriminants. Thus, the curves (62) represent elliptic curves. Finally, by the result of Baker's Theorem and its best improvement by Hajdu and Herendi, the number of the integral points of the curves (58) or (62) is finite. Hence, these points can be effectively determined using the techniques mentioned earlier. The only problem that may appear here is that there is no known algorithm to determine the rank and generators of the Mordell-Weil groups of elliptic curves, there are techniques that work well in practice but there is no guarantee to succeed. If we have such an elliptic curve, then we may follow the previously mentioned argument of Alekseyev and Tengely. As a result, the number of the solutions (n, m, g, B, l) is finite, and they can be effectively determined. This completes the proof of Theorem 2.10. □

PROOF OF THEOREM 2.11. We split the proof of this theorem into two cases regarding the sequences of Fibonacci numbers and Pell numbers in which they satisfy equation (48). The proof of Fibonacci case is divided into two subcases: the first one is if $l = 1$ in which we use the result of Matveev on linear forms in three logarithms and the result of Dujella and Pethő on the method of Baker-Davenport reduction, and the other is the case of $l = 2, 4$ in which we apply our approach presented in Theorem 2.10. On the other hand, the proof of the Pell case will be handled in a similar way using only the result of Theorem 2.10.

The Fibonacci case: $G_n = F_n$**.**

(a) For $l = 1$:

- Step 1. Finding a bound for n in the equation

$$F_n = B \cdot \left(\frac{g^m - 1}{g - 1}\right), \tag{63}$$

where $2 \leq g \leq 9, 1 \leq B \leq \min\{10, g - 1\}$ and $m \geq 2$. Note that since $(g - 1) < 10$ for all $2 \leq g \leq 9$, here we only use the range $1 \leq B \leq (g - 1)$. Suppose that $n > 50$. By substituting Binet's Fibonacci numbers formula (51) in equation (63), we get that

$$\frac{\alpha^n - \beta^n}{\sqrt{5}} = B \cdot \left(\frac{g^m - 1}{g - 1}\right),$$

which can be further written as

$$\alpha^n - \left(\frac{\sqrt{5}B}{g - 1}\right)g^m = \beta^n - \left(\frac{\sqrt{5}B}{g - 1}\right). \tag{64}$$

By taking the absolute value for both sides of the latter equation, we obtain that

$$\left|\alpha^n - \left(\frac{\sqrt{5}B}{g - 1}\right)g^m\right| \leq \alpha^{-50} + \sqrt{5} < 2.3 \tag{65}$$

as $\beta = \frac{-1}{\alpha}, n > 50$ and $B \leq (g - 1)$. Define

$$\Lambda = \log\left(\frac{\sqrt{5}B}{g - 1}\right) - n\log(\alpha) + m\log(g). \tag{66}$$

Since $e^{\Lambda} = \left(\frac{\sqrt{5}B}{g-1}\right)\alpha^{-n}g^m$, we get (from inequality (65)) that

$$\left|e^{\Lambda} - 1\right| < \frac{2.3}{\alpha^n} < \alpha^{-n+2}. \tag{67}$$

To apply Lemma 2.8, we first state and prove the following claim:

Claim: Suppose that Λ is defined in equation (66), then $\Lambda > 0$.

Proof: From equation (64) and the fact that $\beta = \frac{-1}{\alpha}$, we deduce that

$$1 - e^{\Lambda} = \frac{1}{\alpha^n}\Big(\beta^n - \frac{\sqrt{5}B}{g-1}\Big) = \frac{1}{\alpha^n}\Big((-1)^n\alpha^{-n} - \frac{\sqrt{5}B}{g-1}\Big). \tag{68}$$

Now, we consider the following cases regarding the values of n.

- If n is even, then equation (68), with the hypotheses: $-n < -50$, $g \le 9$ and $B \ge 1$, implies that

$$\begin{aligned} 1 - e^{\Lambda} &= \frac{1}{\alpha^n}\Big(\alpha^{-n} - \frac{\sqrt{5}B}{g-1}\Big) \\ &< \frac{1}{\alpha^n}\Big(\alpha^{-50} - \frac{\sqrt{5}}{8}\Big) < 0, \end{aligned}$$

  which leads to $\Lambda > 0$.
- If n is odd, then for all $n > 50, 2 \le g \le 9$ and $1 \le B \le (g-1)$ we have that $\Big((\alpha^{-1})^n + \frac{\sqrt{5}B}{g-1}\Big) > 0$. Therefore, equation (68) again gives

$$1 - e^{\Lambda} = \frac{-1}{\alpha^n}\Big((\alpha^{-1})^n + \frac{\sqrt{5}B}{g-1}\Big) < 0,$$

  which also implies that $\Lambda > 0$.

Thus, the claim is completely proved.

From (67) and the fact that $\Lambda > 0$, we obtain that $\Lambda < e^{\Lambda} - 1 < \alpha^{-n+2}$. Therefore,

$$\log|\Lambda| < (-n+2)\log(\alpha). \tag{69}$$

With respect to the notation of Lemma 2.8 and by comparing equations (54) and (66), we have that

$$\alpha_1 = \Big(\frac{\sqrt{5}B}{g-1}\Big), \alpha_2 = \alpha, \alpha_3 = g, a_1 = 1, a_2 = -n \text{ and } a_3 = m.$$

We also note that the number field $\mathbb{Q}(\alpha_1, \alpha_2, \alpha_3) = \mathbb{Q}(\sqrt{5})$ is of degree $d = 2$. Furthermore, the conjugates of α_1, α_2 and α_3 are $\alpha_1' = -\alpha_1$, $\alpha_2' = \beta$ and $\alpha_3' = \alpha_3$, respectively. Clearly, the minimal polynomial of α_1 is

$$(x - \alpha_1)(x - \alpha_1') = x^2 - \frac{5B^2}{(g-1)^2},$$

which is a divisor of $(g-1)^2x^2 - 5B^2$. Therefore, by using the definition of the logarithmic height of algebraic numbers in (53) we get the logarithmic height of the 2-degree algebraic number α_1 is

$$h(\alpha_1) \leq \frac{1}{2}(2\log(8) + 2\log(\sqrt{5})) < 2.89$$

as $g \leq 9$ and $B \leq (g-1)$. Similarly, $h(\alpha_2) = \frac{\log(\alpha)}{2} < 0.25$ and $h(\alpha_3) = \log(g) < 2.2$ as $g \leq 9$. Hence, from inequality (55) we take $A_1 = 5.78, A_2 = 0.5$ and $A_3 = 4.4$.

Furthermore, to obtain an estimated value for B' using inequality (56), let us first consider and prove the following claim:

<u>Claim:</u> The values of n and m in equation (63) satisfy that $n > m$.

<u>Proof:</u> First of all, from equation (63) we have that

$$F_n \geq \left(\frac{g^m - 1}{g-1}\right) \quad \text{as} \quad B \geq 1. \tag{70}$$

On the other hand, we see that

$$\begin{aligned}\left(\frac{g^m - 1}{g-1}\right) - g^{m-1} &= g^m \cdot \left(\frac{1}{g-1} - \frac{1}{g}\right) - \left(\frac{1}{g-1}\right) \\ &= g^m \cdot \left(\frac{1}{g(g-1)}\right) - \left(\frac{1}{g-1}\right) \\ &> 0 \quad \text{as} \quad m > 1,\end{aligned}$$

which gives

$$\left(\frac{g^m - 1}{g-1}\right) > g^{m-1}. \tag{71}$$

Combining the inequalities (70), (71) and (52), we get that

$$g^{m-1} < F_n \leq \alpha^{n-1}. \tag{72}$$

Taking the logarithm for both sides, we obtain $(m-1)\log(g) < (n-1)\log(\alpha)$, which leads to

$$m < (n-1)\frac{\log(\alpha)}{\log(g)} + 1. \tag{73}$$

Since $\frac{\log(\alpha)}{\log(g)} < 1$ as $g \geq 2$, we get that $m < (n-1) + 1 = n$ with $m \geq 2$. This proves the claim.

Therefore, since $n > 50$ we have that

$$\max\{1, \max\{|a_j|\, A_j/A_1 : 1 \leq j \leq 3\}\} = \max\left\{\frac{0.5}{5.78}n, \frac{4.4}{5.78}m\right\},$$

and then it suffices to take $B' = \frac{5}{6}n$ as $n > m$. From (57), we get that $C_1 < 4.45 \cdot 10^9$ since $\chi = 1$ and $d = 2$. Therefore, Lemma 2.8 yields

$$\log|\Lambda| > -2.27 \cdot 10^{11} \log(11.51n). \tag{74}$$

Combining inequalities (69) and (74), we obtain that

$$2.27 \cdot 10^{11} \log(11.51n) > (n-2)\log(\alpha),$$

which implies that $n < 10^{14}$.

- Step 2. Finding a bound for m in equation (63). For that, we first give the following lemma:

LEMMA 2.12. *The solutions of equation* (63) *satisfy*

$$(n-2)\frac{\log(\alpha)}{\log(g)} < m < (n-1)\frac{\log(\alpha)}{\log(g)} + 1. \tag{75}$$

PROOF. The proof follows easily from combining the fact (52) and equation (63). Indeed, for all $2 \le g \le 9, 1 \le B \le (g-1)$ and $m > 1$, one can see that

$$\alpha^{n-2} \le F_n < g^m.$$

Taking the logarithm for both sides, we obtain that

$$(n-2)\log(\alpha) < m\log(g),$$

which leads to

$$(n-2)\frac{\log(\alpha)}{\log(g)} < m. \tag{76}$$

The upper bound follows from inequality (73). Hence, Lemma 2.12 is proved.

Thus, from the upper bound of inequality (75) and the estimate of n (that is $n < 10^{14}$) we obtain that

$$m < (10^{14}-1)\frac{\log(\alpha)}{\log(g)} + 1 < 7 \cdot 10^{13} \quad \text{as} \quad g \ge 2.$$

- Step 3. Reducing the obtained bounds. We know that $0 < \Lambda < \alpha^{-n+2}$. From inequality (72), we also know that $\alpha^{n-1} > g^{m-1}$, which leads to

$$\alpha^{-n+2} < g^{-m+1}\alpha < g^{-m+2}$$

as $\alpha < g$ for all $2 \le g \le 9$. Hence,

$$0 < m\log(\alpha_3) - n\log(\alpha_2) + \log(\alpha_1) < g^{-m+2}.$$

Dividing the latter inequality by $\log(\alpha_2)$, we get that

$$0 < m\frac{\log(\alpha_3)}{\log(\alpha_2)} - n + \frac{\log(\alpha_1)}{\log(\alpha_2)} < 3 \cdot g^2 \cdot g^{-m}. \tag{77}$$

Without loss of generality and to be more precise, since $\alpha_1 = \left(\frac{\sqrt{5}B}{g-1}\right)$, $\alpha_2 = \alpha$ and $\alpha_3 = g$ for all $g \in \{2, 3, \ldots, 9\}$ and $1 \le B \le (g-1)$, we respectively use the notation "n_g, m_g, B_g" instead of "n, m, B" for the rest of the proof of the bounds reduction step. Therefore, we rewrite (77) in the form

$$0 < m_g\kappa_g - n_g + \mu_g < 3\cdot g^2\cdot g^{-m_g}, \tag{78}$$

where $\kappa_g = \frac{\log(g)}{\log(\alpha)}$ and $\mu_g = \frac{\log\left(\frac{\sqrt{5}B_g}{g-1}\right)}{\log(\alpha)}$. It is clear that $\mu_g \geq \frac{\log\left(\frac{\sqrt{5}}{g-1}\right)}{\log(\alpha)}$ as $B_g \geq 1$. Since α and g are multiplicatively independent, we have κ_g is irrational. Thus, we may denote $\frac{P_{(k,g)}}{Q_{(k,g)}}$ be the k^{th} convergent of the continued fraction of κ_g. Now, we use Lemma 2.9 to reduce the upper bound of m_g (which is very large since $m_g < 7\cdot 10^{13}$). That will lead to reduce the upper bound of n_g. Therefore, we take $M = M_g = 7\cdot 10^{13}$. Moreover, if the conditions of Lemma 2.9 are satisfied; that are, if $Q_{(k,g)} > 6M$ and $\epsilon_g = \|\mu_g Q_{(k,g)}\| - M\|\kappa_g Q_{(k,g)}\| > 0$, then we take $A_g = 3\cdot g^2$ and $K_g = g$. For $g = 2$, we have $B_2 = 1, \kappa_2 = \frac{\log(2)}{\log(\alpha)}$ and $\mu_2 \geq \frac{\log(\sqrt{5})}{\log(\alpha)}$. Therefore, we obtain that

$$\frac{P_{(32,2)}}{Q_{(32,2)}} = \frac{2683806884597620}{1863211227378077}$$

of which we have $Q_{(32,2)} > 6M$ and

$$\epsilon_2 \geq \|\frac{\log(\sqrt{5})}{\log(\alpha)}Q_{(32,2)}\| - M\|\frac{\log(2)}{\log(\alpha)}Q_{(32,2)}\| > 0.4 > 0.$$

Therefore, we have that $A_2 = 3\cdot 2^2$ and $K_2 = 2$. Hence, Lemma 2.9 tells us that there is no solution to inequality (78) (and then to the Diophantine equation (63) in case of $g = 2$; that is,

$$F_{n_2} = (2^{m_2} - 1)) \tag{79}$$

in the range

$$\left[\left\lfloor\frac{\log(A_2Q_{(32,2)}/\epsilon_2)}{\log(K_2)}\right\rfloor + 1, M\right] = [56, 7\cdot 10^{13}].$$

Therefore, $m_2 \leq 56$ and inequality (76) gives us $n_2 < 83$. To finish, we use SageMath [**231**] to print all the Fibonacci numbers in the range $50 < n_2 < 83$ of which we see that there are no Fibonacci numbers satisfying equation (79) with $2 \leq m_2 \leq 56$. However, in the range $3 \leq n_2 \leq 50$ we get the solution $(n_2, m_2, g, B_2, l) = (4, 2, 2, 1, 1)$. Let us now consider the case $g = 6$, which implies that $1 \leq B_6 \leq 5, \kappa_6 = \frac{\log(6)}{\log(\alpha)}$ and $\mu_6 \geq \frac{\log(1/\sqrt{5})}{\log(\alpha)}$. Thus, $Q_{(30,6)} = 1232281049712607 > 6M$ and $\epsilon_6 > 0.1 > 0$. For that, we take $A_6 = 3\cdot 6^2$ and $K_6 = 6$, and Lemma 2.9

leads to the unsolvablity of the equation

$$F_{n_6} = \frac{B_6}{5} \cdot (6^{m_6} - 1), \tag{80}$$

with $24 \leq m_6 \leq M$. Therefore, $m_6 \leq 24$, which implies that $n_6 < 92$. Again, we get no solutions to equation (80) (in fact, to equation (63) in case of $g = 6$) with $50 < n_6 < 92$, but we get the solution $(8, 2, 6, 3, 1)$ where $3 \leq n_6 \leq 50$. In a similar way, for all the remaining values of g, one can show that the hypotheses of Lemma 2.9 are satisfied and determine the other desired solutions in the theorem in case of $l = 1$.

(b) For $l \in \{2, 4\}$:

- If $l = 2$, then quation (48) becomes

$$F_n = B \cdot \left(\frac{g^{2m} - 1}{g^2 - 1}\right). \tag{81}$$

We consider in detail the case where we have $g = 2$, and the remaining values of g will be pursued in a similar way following the proof of Theorem 2.10. Since $P = 1, Q = -1, l = 2, t = 1$ and $g = 2$, we have that $D_2 = 5, G = 3$ and $1 \leq B \leq 3$. Therefore, for all $B \in \{1, 2, 3\}$, equation (60) leads to the biquadratic elliptic curves

$$y^2 = 5x^4 - 10x^2 + 41, \tag{82}$$

$$y^2 = 20x^4 - 40x^2 + 56, \tag{83}$$

$$y^2 = 45x^4 - 90x^2 + 81, \tag{84}$$

and also equation (61) gives the curves

$$y^2 = 5x^4 - 10x^2 - 31, \tag{85}$$

$$y^2 = 20x^4 - 40x^2 - 16, \tag{86}$$

$$y^2 = 45x^4 - 90x^2 + 9, \tag{87}$$

respectively, where $x = 2^m$ and $y = 3L_n$. Let us now consider the curve (82) and by using the previously mentioned Magma function `SIntegralLjunggrenPoints()`, we get the following integral points with positive values for the x−coordinates

$$[[1, -6], [2, 9], [5, 54], [8, -141]].$$

Combining the values of x of these integral points with $x = 2^m$, we only obtain $m = 3$. Therefore, since $B = 1, g = 2$ and $m = 3$, equation (81) implies that $n = 8$. Hence, we get the solution $(n, m, g, B, l) = (8, 3, 2, 1, 2)$. Next, we consider (83), which has no integral points other than $[x, y] = [1, 6]$ in which the value of x is positive. Thus, we have no solution for equation (81). Similarly, we get no solution for (81) in case of equations (84), (86) and (87). Finally, we deal with equation

(85) and here we get $x = 4$, which leads to $m = 2$. Hence, we get the solution $(5, 2, 2, 1, 2)$. The other remaining values of g can be treated in a similar way. Indeed, we get only one solution, which is $(9, 2, 4, 2, 2)$ in case of having $B = 2$ and $g = 4$.

- If $l = 4$, then quation (48) implies that

$$F_n = B \cdot \Big(\frac{g^{4m} - 1}{g^4 - 1}\Big). \tag{88}$$

In fact, it can be proven completely following the same approach in the previous case in which we had $l = 2$. But, let us treat this case using the elliptic curve equation (62). For that, we may consider the case where we have $g = 2$ and $B = 2$. Again, since $D = 5, t = 2$ and $G = 15$, equation (62) gives the curves

$$Y^2 = X^3 - 40X^2 - 17600X, \tag{89}$$

$$Y^2 = X^3 - 40X^2 + 18400X, \tag{90}$$

where $X = 20(16)^m$ and $Y = 300(4)^m L_n$. By considering equation (89) and using the SageMath function `integral_points()`, we get the integral points

$$[(176 : 1056 : 1), (220 : 2200 : 1), (225 : 2325 : 1),$$
$$(320 : 4800 : 1), (529 : 11293 : 1), (4400 : 290400 :$$
$$1), (5120 : 364800 : 1), (818620 : 740649800 : 1)]$$

in which we considered only the points with positive values for the $X-$coordinates. From these points, only $X = 5120$ leads to a solution of equation (88), that is $(n, m, g, B, l) = (9, 2, 2, 2, 4)$. On the other hand, the integral points of the elliptic curve (90) give no solution to equation (88). Furthermore, the other remaining cases for all the values of g can be handled in a similar way. More precisely, one can show that the equation (88) has no more solutions.

The Fibonacci case is completely proved.

The Pell case: $G_n = P_n$**.**

Here, we have $l = 2$. Thus, equation (48) becomes

$$P_n = B \cdot \Big(\frac{g^{2m} - 1}{g^2 - 1}\Big). \tag{91}$$

Solving this equation completely is handled in a similar way in case of Fibonacci numbers with l is even. Here, we have $D_2 = 8$ and $t = 1$. If we consider $g = 2$ and $B = 1$, then we get $G = 3$. These lead to the biquadratic elliptic curves

$$y^2 = 8x^4 - 16x^2 - 28, \tag{92}$$

$$y^2 = 8x^4 - 16x^2 + 44, \tag{93}$$

where $x = 2^m$ and $y = 3Q_n$ such that Q_n denotes the general term of the sequence of Pell-Lucas numbers that is defined in (26). Equation (92) leads to the solution $(n, m, g, B, l) = (3, 2, 2, 1, 2)$, and equation (93) implies no solution to equation (91). The remaining cases are treated similarly. As a result, equation (91) does not have any more solutions. Hence, this case is also proved.

Therefore, Theorem 2.11 is completely proved. $\square$

CHAPTER 3

Diophantine equations of the form $G(X,Y,Z) := AX^2 + BY^r + CZ^2$ involving linear recurrence sequences

3.1. Solutions of the Diophantine equation $7X^2 + Y^7 = Z^2$ from linear recurrence sequences

Consider the Diophantine equation

$$AX^2 + BY^r = C'Z^2, \tag{94}$$

where A, B, C' and r are nonzero integers such that $r > 1$. According to the result of Beukers [**22**] mentioned in Theorem 1.12 related to the Fermat-Catalan Diophantine equation, we have that equation (94) has either no solution or infinitely many relatively prime integer solutions (X, Y, Z) (see also, e.g. [**67**] or [**176**]). Indeed, according to Mordell [**176**, page 111] equation (94) has infinitely many integer solutions if $B = 1$ and r is odd.

If $(F_n)_{n\geq 0}$ and $(L_n)_{n\geq 0}$ are the sequences of Fibonacci and Lucas numbers (which are defined by (23) and (25), respectively), in this section we present a technique with which we can investigate the nontrivial integer solutions (X, Y, Z) of equations of the form

$$AX^2 + Y^r = C'Z^2,$$

where A, C' and r are certain nonzero integers with $r > 1$ being odd and $(X, Y) = (L_n, F_n)$(or $(X, Y) = (F_n, L_n)$). We also remark that this technique can be applied on such equations for which they satisfy some conditions, that will be mentioned later along a procedure presented by Kedlaya in [**128**] (i.e. Kedlaya's procedure in 3.1.1.1). We indeed present the use of this technique for studying such special solutions of the Diophantine equation

$$7X^2 + Y^7 = Z^2. \tag{95}$$

Based on the parametrizations of the solutions of equation (14) (according to [**176**, page 111]), the integer solutions of the above equation can be parametrized as

$$\begin{aligned}
X &= 7a_1^6b_1 + 245a_1^4b_1^3 + 1029a_1^2b_1^5 + 343b_1^7,\\
Y &= a_1^2 - 7b_1^2,\\
Z &= a_1^7 + 147a_1^5b_1^2 + 1715a_1^3b_1^4 + 2401a_1b_1^6,
\end{aligned}$$

where a_1 and b_1 are arbitrary integers, which provide infinitely many integer solutions of equation (95). More precisely, in this section we deal with special solutions of this equation, namely where $(X,Y) = (L_n, F_n)$(or $(X,Y) = (F_n, L_n)$). By using the identity relationship between the terms of the Fibonacci and Lucas sequences derived from equation (21), i.e. $L_n^2 - 5F_n^2 = \pm 4$, these solutions are clearly equivalent to the solutions of the systems

$$L_n^2 - 5F_n^2 = \pm 4, \quad 7L_n^2 + F_n^7 = Z^2$$

and

$$L_n^2 - 5F_n^2 = \pm 4, \quad L_n^7 + 7F_n^2 = Z^2.$$

In other words, we examine the solutions to the following systems of Diophantine equations

$$x^2 - 5y^2 = \pm 4, \quad 7x^2 + y^7 = z^2, \tag{96}$$

$$x^2 - 5y^2 = \pm 4, \quad x^7 + 7y^2 = z^2, \tag{97}$$

where $x = L_n$, $y = F_n$ and $z = Z$ is a nonzero integer. A solution (x,y,z) of any system in (96) or (97) represents a solution (x,y) of one of its special Pell equations with the restriction given by the corresponding equation.

Historically, several authors investigated the existence and nonexistence of the integer solutions of certain systems of Diophantine equations of the form

$$x^2 - ay^2 = b, \quad P(x,y) = z^2, \tag{98}$$

where a is a positive integer that is not a perfect square, b is a nonzero integer and $P(x,y)$ is a polynomial with integer coefficients. Many of the studies related to systems of the form (98) use Baker's results on linear forms in logarithms of algebraic numbers [**10**] to give an upper bound on the size of the solutions. Using this bound with some techniques of Diophantine approximation, Baker and Davenport [**13**] proved that there is no solution in nonnegative integers other than $(x,y,z) = (1,1,1)$ or $(19,11,31)$ for the system

$$x^2 - 3y^2 = -2, \quad z^2 - 8y^2 = -7.$$

Brown [**39**] proved that the equations

$$y^2 - 8t^2 = 1, \quad u^2 - 5t^2 = 1$$

have no common solution other than $(y,t,u) = (1,0,1)$ using Grinstead's technique in [**96**]. Szalay [**236**] presented an alternative procedure for solving systems of simultaneous Pell equations

$$a_1x^2 + b_1y^2 = c_1, \quad a_2x^2 + b_2z^2 = c_2$$

in nonnegative integers x, y and z, with relatively small coefficients. He implemented the algorithm of this procedure in Magma to verify famous examples and give a new theorem related to such systems. In general, one can guarantee the finiteness of the number of solutions of (98) by the work of Thue [**246**] or Siegel [**216**]. On the other hand, many authors have given elementary solutions to systems of the form (98) such as Cohn [**57**] who considered the case where P is a linear polynomial. Cohn's method uses congruence arguments to eliminate some cases and a clever invocation of quadratic reciprocity to handle the remaining cases. The congruence arguments are very sufficient if there exists no solution in such a system, however they fail in the presence of a solution. Mohanty and Ramasamy [**171**] adapted this method to show that the system of equations

$$x^2 - 5y^2 = -20, \quad z^2 - 2y^2 = 1$$

has no solution other than $(x, y, z) = (0, 2, 3)$. Muriefah and Al Rashed [**1**] showed that the system

$$y^2 - 5x^2 = 4, \quad z^2 - 442x^2 = 441$$

has no integer solutions using a similar method to that presented by Mohanty and Ramasamy. Additionally, Peker and Cenberci [**182**] proved that the system

$$y^2 - 10x^2 = 9, \quad z^2 - 17x^2 = 16$$

can not be solved simultaneously in nonzero integers x, y, z using the same method with Muriefah and Rashed. Kedlaya [**128**] gave a general procedure, based on the methods of Cohn and the theory of Pell equations, that solves many systems of the form (98). In fact, he applied this approach on several examples in which P is univariate with degree at most two. Moreover, in some cases this procedure fails to solve a system completely. To investigate the solutions of the Diophantine equation (95) from the sequences of Fibonacci numbers and Lucas numbers, i.e. $(X, Y) = (L_n, F_n)$ (or $(X, Y) = (F_n, L_n)$), we use Kedlaya's procedure and similar techniques adapted by the methods of Mohanty and Ramasamy, Muriefah and Rashed, and Peker and Cenberci to determine and prove whether or not each of the four systems of equations in (96) and (97) has a solution. We employ Kedlaya's procedure and the techniques of using the congruence arguments and the quadratic reciprocity to prove that the system

$$x^2 - 5y^2 = 4, \quad 7x^2 + y^7 = z^2$$

has no more solutions other than $(x, y, z) = (3, 1, \pm 8)$, and each of the other three systems can not be solved simultaneously.

Next, we introduce some auxiliary results for which we need to prove our main results.

3.1.1. Auxiliary results.

For the proofs of our theorems, we need the following Lemma 3.1 presented by Copley [**60**], Lemmas 3.2, 3.3 and 3.4 and a procedure presented by Kedlaya [**128**] for checking if a given list of solutions to a system of the form (98) is complete, and a remark, i.e. Remark 3.5, that shows the general forms of nonnegative solutions for the Pell type equations

$$x^2 - 5y^2 = \pm 4.$$

LEMMA 3.1. *Let* $(x_k + y_k\sqrt{a})(k = 0, 1, 2, 3, \ldots)$ *be the solution of* $x^2 - ay^2 = b$ *in a fixed class* C*, where* b *is a given nonzero integer and* a *is a positive integer which is not a square, then*

$$x_{-k} = x_k, \qquad y_{-k} = -y_k, \tag{99}$$

$$x_{k+r} = u_r x_k + a v_r y_k, \tag{100}$$

$$y_{k+r} = u_r y_k + v_r x_k, \tag{101}$$

where $(u_r + v_r\sqrt{a}) = (u_1 + v_1\sqrt{a})^r$ *such that* (u_1, v_1) *is the fundamental solution of the Pell equation* $u^2 - av^2 = 1$.

LEMMA 3.2. *For all* k, ω, r *we have* $y_{k+2\omega r} \equiv (-1)^\omega y_k \pmod{u_r}$ *and* $y_{k+2\omega r} \equiv y_k \pmod{v_r}$ *(of course, the same result holds for* u_k, v_k *or* x_k *as well).*

LEMMA 3.3. *For all* k, ω *we have* $v_k | v_{\omega k}$*; if* ω *is odd, we also have* $u_k | u_{\omega k}$.

LEMMA 3.4. *If the sequence* $\{f_k\}$ *satisfies the recurrence relation*

$$f_{k+1} = 2 f_k u_1 - f_{k-1},$$

then for any positive integer χ, $\{f_k \pmod{\chi}\}$ *is completely periodic (of course, the same result holds for* $f_k = u_k, v_k, x_k$ *or* y_k *as well).*

3.1.1.1. *Kedlaya's procedure.*

Denote by (u_k, v_k) be the k^{th} solution of the Pell equation

$$u^2 - av^2 = 1.$$

For each base solution (x_0, y_0) of the equation $x^2 - ay^2 = b$, let S be the set of integers m such that (x_m, y_m) is in the given list of solutions. One can prove that $P(x_m, y_m)$ is a prefect square if and only if $m \in S$ as follows (without having to give up):

- For each $m \in S$, let $\alpha = P(-x_m, -y_m)$.
- If $|\alpha|$ is a perfect square, we give up; otherwise, let β be the product of all the primes that divide α an odd number of times.
- Let l be the period of $\{u_k (\mathrm{mod}\ \beta)\}$ (the period is guaranteed by Lemma 3.4) and d be the largest odd divisor of l.
- Let q be the largest integer such that $2^q | l$, unless 4 does not divide l, in which case let $q = 2$.
- Let s be the order of 2 in the group $(\mathbb{Z}/d\mathbb{Z})^\times$.
- Define the set $U = \{t \in \{0, \ldots, d-1\} : \left(\frac{u_{2^q t}}{\beta}\right) = -1\}$.
- If U is empty, we give up; otherwise find an odd number j such that for each $\epsilon = q, \ldots, q+s-1$, there exist $t \in U$ and $g \mid j$ with $2^{\epsilon - q} g \equiv t \pmod{\beta}$.
- Let $\gamma_m = 2^q j$ and γ be twice the least common multiple of γ_m for all $m \in S$.
- Find an integer δ with the following property: for every $k \in \{0, \ldots, \delta\gamma - 1\}$, either $k \equiv m \pmod{2\gamma_m}$ for some $m \in S$; or there exists a prime number p such that $P(x_k, y_k)$ is a nonresidue $\pmod{p}$, with $\{x_i \pmod{p}\}$ and

$\{y_i \pmod p\}$ have periods dividing $\delta\gamma$. Using Lemmas 3.2 and 3.3, we note that the period condition can be guaranteed by having $p|v_\kappa$ for some κ, where $2\kappa|\delta\gamma$.

- Suppose that δ can be found satisfying the specified properties. To show that $P(x_m, y_m)$ is a prefect square if and only if $m \in S$, assume that there exists $k \notin S$ such that $P(x_k, y_k)$ is a perfect square. By the construction of δ, there exists m such that $k \equiv m \pmod{2\gamma_m}$, or else there exists a prime number p such that $P(x_k, y_k)$ is a nonresidue $\pmod p$. Since $k \notin S$, so $k \neq m$ and $k = m + 2^{\epsilon+1}jh$ for some h, ϵ with h odd and $\epsilon \geq q$. Using Lemma 3.2, we get that

$$x_k \equiv -x_m \pmod{u_{j2^\epsilon}} \quad \text{and} \quad y_k \equiv -y_m \pmod{u_{j2^\epsilon}}.$$

Therefore,

$$P(x_k, y_k) \equiv P(-x_m, -y_m) = \alpha \pmod{u_{j2^\epsilon}}.$$

The construction gives that for some $t \in U$ and some $g \mid j$ with $2^{\epsilon-q}g \equiv t \pmod \beta$. It is clear that $\epsilon \geq q \geq 2$ and $\{u_k \pmod 8\}$ has period dividing 4. Thus, the Jacobi symbols $\left(\frac{-1}{u_{2^\epsilon g}}\right)$ and $\left(\frac{2}{u_{2^\epsilon g}}\right)$ both equal 1. Since $|\alpha|/\beta$ is a perfect square and $u_{g2^\epsilon}|u_{j2^\epsilon}$ by Lemma 3.3, we have by quadratic reciprocity

$$\left(\frac{P(x_k,y_k)}{u_{2^\epsilon g}}\right) = \left(\frac{\alpha}{u_{2^\epsilon g}}\right) = \left(\frac{\beta}{u_{2^\epsilon g}}\right) = \left(\frac{u_{2^\epsilon g}}{\beta}\right) = \left(\frac{u_{2^q t}}{\beta}\right) = -1,$$

which contradicts the assumption that $P(x_k, y_k)$ is a perfect square.

REMARK 3.5. *The Pell equation $u^2 - 5v^2 = 1$ has the fundamental solution $(u_1, v_1) = (9, 4)$, and the Pell type equation $x^2 - 5y^2 = 4$ has three non associated classes of solutions with the fundamental solutions $3 + \sqrt{5}, 3 - \sqrt{5}$ and 2. Therefore, its general solutions are given by*

$$x_k + y_k\sqrt{5} = (3 + \sqrt{5})(9 + 4\sqrt{5})^k, \tag{102}$$

$$x_k + y_k\sqrt{5} = (3 - \sqrt{5})(9 + 4\sqrt{5})^k, \tag{103}$$

$$x_k + y_k\sqrt{5} = (2)(9 + 4\sqrt{5})^k, \tag{104}$$

respectively. Similarly, the general solutions of the Pell type equation $x^2 - 5y^2 = -4$ are given by

$$x_k + y_k\sqrt{5} = (1 + \sqrt{5})(9 + 4\sqrt{5})^k, \tag{105}$$

$$x_k + y_k\sqrt{5} = (-1 + \sqrt{5})(9 + 4\sqrt{5})^k, \tag{106}$$

$$x_k + y_k\sqrt{5} = (4 + 2\sqrt{5})(9 + 4\sqrt{5})^k, \tag{107}$$

respectively. For more details about the Pell type equations, see Subsection 1.2.1 of Chapter 1.

3.1.2. New results.

In the following we present our main results related to the solutions of the Diophantine equation (95) in case of $(X,Y) = (L_n, F_n)$(or $(X,Y) = (F_n, L_n)$) and Z is a nonzero integer. These results consist of two theorems, which are respectively represented as Theorem 1 and Theorem 2 in [**104**].

THEOREM 3.6. *Suppose that $X = L_n$ and $Y = F_n$, then the Diophantine equation* (95) *has no more solutions other than $(X,Y,Z) = (3,1,\pm 8)$.*

THEOREM 3.7. *The Diophantine equation* (95) *has no solutions in integers X, Y and Z if $X = F_n$ and $Y = L_n$.*

3.1.3. Proofs of the results.

PROOF OF THEOREM 3.6. To prove this theorem, we have to show that $(3,1,8)$ and $(3,1,-8)$ are the only solutions to the systems of the simultaneous Diophantine equations in (96). In fact, they are the only solutions to the system

$$x^2 - 5y^2 = 4, \tag{108}$$

$$7x^2 + y^7 = z^2, \tag{109}$$

where $x = L_n, y = F_n$ and $z = Z$. Now, let $P(x,y) = 7x^2 + y^7$. Considering equation (102) and using Kedlaya's procedure described in 3.1.1.1, it is possible to show that $P(x_m, y_m)$ is a perfect square if and only if $m \in S = \{0\}$ and the set

$$\{(x_0, y_0, z)\} = \{(3,1,-8), (3,1,8)\}$$

is a complete list of solutions to the system of the Diophantine equations (108) and (109) with the procedure's output: $\alpha = \beta = 62, l = d = 5, q = 2, s = 3, U = \{2,3\}, \gamma_m = 60, \gamma = 120$ and $\delta = 1$ such that for $k = 0$, $k \equiv m \equiv 0 \pmod{120}$. Following the last step in the procedure, one can easily show that there exists no k other than $k = 0$ such that $k \equiv 0 \pmod{120}$ and $P(x_k, y_k)$ is a perfect square. Assume, for the sake of contradiction, that there exists $k \notin S$ such that $k \equiv 0 \pmod{120}$ and $P(x_k, y_k)$ is a perfect square. Therefore, $k = 2^{\epsilon+1} jh = 2^{\epsilon+1} 15h$ for some h, ϵ with h odd and $\epsilon \geq q = 2$. Using Lemma 3.2, we obtain

$$x_k \equiv -x_0 = -3 \pmod{u_{2^\epsilon 15}} \quad \text{and} \quad y_k \equiv -y_0 = -1 \pmod{u_{2^\epsilon 15}},$$

which imply that $P(x_k, y_k) \equiv P(-3,-1) = 62 = \alpha \pmod{u_{2^\epsilon 15}}$. Since

$$2^{\epsilon - q} g \equiv t \pmod{\beta}$$

for some $t \in U = \{2,3\}$ and some $g \mid 15$ and $|\alpha|/\beta$ is equal 1 which is a perfect square, we get that $u_{2^\epsilon g} | u_{2^\epsilon 15}$ by Lemma 3.3. Moreover, we have the Jacobi symbol $\left(\frac{2}{u_{2^\epsilon g}}\right)$ is equal 1. Therefore, we obtain by the quadratic reciprocity that

$$\left(\frac{P(x_k, y_k)}{u_{2^\epsilon g}}\right) = \left(\frac{62}{u_{2^\epsilon g}}\right) = \left(\frac{u_{2^\epsilon g}}{62}\right) = \left(\frac{u_{2^2 t}}{62}\right) = \left(\frac{37}{62}\right) = -1$$

for all t, contradicting the assumption that $P(x_k, y_k)$ is a perfect square. Next, we consider $k \neq 0$. From equations (100) and (101) in Lemma 3.1, we can write

$$x_{k+15} = (3220013013190122249)x_k + 5(1440033597185408060)y_k, \tag{110}$$
$$y_{k+15} = (3220013013190122249)y_k + (1440033597185408060)x_k, \tag{111}$$

which imply that

$$x_{k+15} \equiv x_k \pmod{11} \quad \text{and} \quad y_{k+15} \equiv y_k \pmod{11}, \tag{112}$$
$$x_{k+15} \equiv -x_k \pmod{17} \quad \text{and} \quad y_{k+15} \equiv -y_k \pmod{17}, \tag{113}$$
$$x_{k+15} \equiv x_k \pmod{19} \quad \text{and} \quad y_{k+15} \equiv y_k \pmod{19}, \tag{114}$$
$$x_{k+15} \equiv 35y_k \pmod{41} \quad \text{and} \quad y_{k+15} \equiv 7x_k \pmod{41}, \tag{115}$$
$$x_{k+15} \equiv -x_k \pmod{61} \quad \text{and} \quad y_{k+15} \equiv -y_k \pmod{61}, \tag{116}$$
$$x_{k+15} \equiv 40y_k \pmod{107} \quad \text{and} \quad y_{k+15} \equiv 8x_k \pmod{107}, \tag{117}$$
$$x_{k+15} \equiv x_k \pmod{181} \quad \text{and} \quad y_{k+15} \equiv y_k \pmod{181}, \tag{118}$$
$$x_{k+15} \equiv x_k \pmod{541} \quad \text{and} \quad y_{k+15} \equiv y_k \pmod{541}, \tag{119}$$
$$x_{k+15} \equiv -x_k \pmod{109441} \quad \text{and} \quad y_{k+15} \equiv -y_k \pmod{109441}, \tag{120}$$
$$x_{k+15} \equiv 4160200y_k \pmod{\xi} \quad \text{and} \quad y_{k+15} \equiv 832040x_k \pmod{\xi}, \tag{121}$$

where $\xi = 10783342081$. From (112), equation (109) becomes

$$z^2 \equiv 7x_k^2 + y_k^7 \pmod{11}.$$

If $k \equiv 1 \pmod{15}$, then $x_k \equiv x_1 \equiv 3 \pmod{11}$ and $y_k \equiv y_1 \equiv 10 \pmod{11}$, which imply that $z^2 \equiv 7 \pmod{11}$, but the Legendre symbol $\left(\frac{7}{11}\right) = -1$. So $k \not\equiv 1 \pmod{15}$. Next, if $k \equiv 3 \pmod{15}$, then $z^2 \equiv 6 \pmod{11}$, which is impossible since $\left(\frac{6}{11}\right) = -1$. Hence, $k \not\equiv 3 \pmod{15}$. From (113), equation (109) implies that

$$z^2 \equiv 7x_k^2 - y_k^7 \pmod{17}.$$

If $k \equiv 4 \pmod{15}$, we get $x_k \equiv x_4 \equiv 4 \pmod{17}$ and $y_k \equiv y_4 \equiv 13 \pmod{17}$. Thus, $z^2 \equiv 6 \pmod{17}$, but $\left(\frac{6}{17}\right) = -1$. Therefore, $k \not\equiv 4 \pmod{15}$. Moreover, if $k \equiv 5 \pmod{15}$ leads to $z^2 \equiv 14 \pmod{17}$, then this gives a contradiction again. Thus, $k \not\equiv 5 \pmod{15}$. Using (115) and from equation (109), we get

$$z^2 \equiv 17x_k^7 + 6y_k^2 \pmod{41}.$$

If $k \equiv 12, 14 \pmod{15}$, we obtain $z^2 \equiv 29 \pmod{41}$. This is impossible since 29 is a quadratic nonresidue modulo 41. Hence, $k \not\equiv 12, 14 \pmod{15}$. From (116), equation (109) gives

$$z^2 \equiv 7x_k^2 - y_k^7 \pmod{61}.$$

Similarly, if $k \equiv 7 \pmod{15}$ or $k \equiv 9 \pmod{15}$, then $z^2 \equiv 43 \pmod{61}$ or $z^2 \equiv 29 \pmod{61}$, respectively. But, these yield a contradiction since $\left(\frac{43}{61}\right) = -1 = \left(\frac{29}{61}\right)$. So $k \not\equiv 7, 9 \pmod{15}$. Finally, using (119), equation (109) implies that

$$z^2 \equiv 7x_k^2 + y_k^7 \pmod{541},$$

which is impossible if $k \equiv 2,6,8,10,11,13 \pmod{15}$. Therefore,

$$k \not\equiv 2,6,8,10,11,13 \pmod{15}.$$

Here, we have proved the completeness of the given list of solutions related to equation (102). Then it remains to show that the equations (108) and (109) have no common solution at the equations (103) and (104) using the above techniques of congruence arguments and the quadratic reciprocity. Now, we consider equation (103). By using (112), we get $z^2 \equiv 7 \pmod{11}$ if $k \equiv 0 \pmod{15}$. However, $\left(\frac{7}{11}\right) = -1$. Also, if $k \equiv 2 \pmod{15}$, we get a contradiction since $z^2 \equiv 6 \pmod{11}$ is impossible. Therefore, $k \not\equiv 0,2 \pmod{15}$. Next, from (114), equation (109) leads to

$$z^2 \equiv 7x_k^2 + y_k^7 \pmod{19}.$$

If $k \equiv 1 \pmod{15}$, then $x_k \equiv x_1 \equiv 7 \pmod{19}$ and $y_k \equiv y_1 \equiv 3 \pmod{19}$. So $z^2 \equiv 3 \pmod{19}$, but this is impossible since 3 is a quadratic nonresidue modulo 19, hence $k \not\equiv 1 \pmod{15}$. From (115), if $k \equiv 10 \pmod{15}$, then $z^2 \equiv 14 \pmod{41}$. This again leads to a contradiction since $\left(\frac{14}{41}\right) = -1$, thus $k \not\equiv 10 \pmod{15}$. Using (116), the equation

$$z^2 \equiv 7x_k^2 - y_k^7 \pmod{61}$$

leads to $z^2 \equiv 51 \pmod{61}, z^2 \equiv 29 \pmod{61}$ or $z^2 \equiv 43 \pmod{61}$ if $k \equiv 4 \pmod{15}, k \equiv 6 \pmod{15}$ or $k \equiv 8 \pmod{15}$, respectively. But, $29, 43$ and 51 are quadratic nonresidues modulo 61, which implies that $k \not\equiv 4,6,8 \pmod{15}$. If we use equation (118), (109) implies that

$$z^2 \equiv 7x_k^2 + y_k^7 \pmod{181}.$$

Here, we face a contradiction if $k \equiv 3,7,9 \pmod{15}$. Therefore,

$$k \not\equiv 3,7,9 \pmod{15}.$$

From (119), the equation

$$z^2 \equiv 7x_k^2 + y_k^7 \pmod{541}$$

and $k \equiv 5,11,12,13,14 \pmod{15}$ yield a contradiction. So

$$k \not\equiv 5,11,12,13,14 \pmod{15}.$$

Finally, we consider (104). If $k \equiv 0 \pmod{15}$, then $z^2 \equiv 11 \pmod{17}$, again giving a contradiction since $\left(\frac{11}{17}\right) = -1$. Moreover, if $k \equiv 5 \pmod{15}$, then $z^2 \equiv 5 \pmod{17}$. This is again impossible, so $k \not\equiv 0,5 \pmod{15}$. Now, we use equation (114). If $k \equiv 1 \pmod{15}$, then $x_k \equiv x_1 \equiv 18 \pmod{19}$ and $y_k \equiv y_1 \equiv 8 \pmod{19}$. This implies that $z^2 \equiv 15 \pmod{19}$, but 15 is a quadratic nonresidue modulo 19. Hence, $k \not\equiv 1 \pmod{15}$. From (118), we get

$$z^2 \equiv 155 \pmod{181}$$

if $k \equiv 2 \pmod{15}$. But, $\left(\frac{155}{181}\right) = -1$. Furthermore, if $k \equiv 3 \pmod{15}$, we obtain $z^2 \equiv 22 \pmod{181}$, again yielding a contradiction. Similarly,

$$k \equiv 4,6,9,10,11,12,13,14 \pmod{15}$$

again leads to a contradiction. So

$$k \not\equiv 2, 3, 4, 6, 9, 10, 11, 12, 13, 14 \pmod{15}.$$

Using equation (119) with $k \equiv 7 \pmod{15}$, we get $z^2 \equiv 502 \pmod{541}$. This is also impossible since $\left(\frac{502}{541}\right) = -1$. Therefore, $k \not\equiv 7 \pmod{15}$. If we use equation (121) for $k \equiv 8 \pmod{15}$, then (109) implies that

$$z^2 \equiv 3401662621 \pmod{10783342081}.$$

This is impossible and hence $k \not\equiv 8 \pmod{15}$. We have thus proved that the equations (108) and (109) have no common solutions other than

$$(x, y, z) = (3, 1, \pm 8) = (L_2, \{F_1, F_2\}, z) = (X, Y, Z).$$

To complete the proof of the theorem, we must show that the other system of the simultaneous Diophantine equations in (96), i.e.

$$x^2 - 5y^2 = -4, \tag{122}$$

$$7x^2 + y^7 = z^2, \tag{123}$$

has no integer solution (x, y, z) such that $x = L_n, y = F_n$ and $z = Z$. Again, we use the same techniques of congruence arguments and the quadratic reciprocity to exhaust all the possibilities of $k \equiv \rho \pmod{r}$ for a proper r and $\rho = 0, 1, 2, \dots, r-1$. From equations (100) and (101), we can write

$$x_{k+10} = (1730726404001)x_k + 5(774004377960)y_k, \tag{124}$$

$$y_{k+10} = (1730726404001)y_k + (774004377960)x_k, \tag{125}$$

which lead to

$$x_{k+10} \equiv x_k \pmod{11} \quad \text{and} \quad y_{k+10} \equiv y_k \pmod{11}, \tag{126}$$

$$x_{k+10} \equiv 15y_k \pmod{23} \quad \text{and} \quad y_{k+10} \equiv 3x_k \pmod{23}, \tag{127}$$

$$x_{k+10} \equiv x_k \pmod{31} \quad \text{and} \quad y_{k+10} \equiv y_k \pmod{31}, \tag{128}$$

$$x_{k+10} \equiv -x_k \pmod{41} \quad \text{and} \quad y_{k+10} \equiv -y_k \pmod{41}, \tag{129}$$

$$x_{k+10} \equiv x_k \pmod{61} \quad \text{and} \quad y_{k+10} \equiv y_k \pmod{61}, \tag{130}$$

$$x_{k+10} \equiv 85y_k \pmod{241} \quad \text{and} \quad y_{k+10} \equiv 17x_k \pmod{241}, \tag{131}$$

$$x_{k+10} \equiv -x_k \pmod{2521} \quad \text{and} \quad y_{k+10} \equiv -y_k \pmod{2521}. \tag{132}$$

First, we consider (105). From (126), equation (123) gives

$$z^2 \equiv 7x_k^2 + y_k^7 \pmod{11}.$$

If $k \equiv 0, 3 \pmod{10}$, then $z^2 \equiv 8 \pmod{11}$. But, 8 is a quadratic nonresidue modulo 11. So $k \not\equiv 0, 3 \pmod{10}$. Using (128), equation (123) implies that

$$z^2 \equiv 7x_k^2 + y_k^7 \pmod{31}.$$

If $k \equiv 2 \pmod{10}$, then $x_k \equiv x_2 \equiv 25 \pmod{31}$ and $y_k \equiv y_2 \equiv 16 \pmod{31}$, which yield $z^2 \equiv 12 \pmod{31}$. This is impossible, hence $k \not\equiv 2 \pmod{10}$. Moreover, if

$k \equiv 4 \pmod{10}$, then $z^2 \equiv 15 \pmod{31}$, again leading to a contradiction. So $k \not\equiv 4 \pmod{10}$. From (130), we get

$$z^2 \equiv 7x_k^2 + y_k^7 \pmod{61}.$$

If $k \equiv 1 \pmod{10}$, then $z^2 \equiv 44 \pmod{61}$. However, $\left(\frac{44}{61}\right) = -1$. This gives $k \not\equiv 1 \pmod{10}$. In a similar way, if $k \equiv 5, 6, 9 \pmod{10}$, we obtain a contradiction again. Therefore, $k \not\equiv 5, 6, 9 \pmod{10}$. From (131), equation (123) leads to

$$z^2 \equiv 23x_k^7 + 206y_k^2 \pmod{241}.$$

If $k \equiv 7 \pmod{10}$ or $k \equiv 8 \pmod{10}$, then $z^2 \equiv 153 \pmod{241}$ or $z^2 \equiv 68 \pmod{241}$, respectively and again giving a contradiction. Hence, $k \not\equiv 7, 8 \pmod{10}$.

Next, we consider (106). From (129), equation (123) gives

$$z^2 \equiv 7x_k^2 - y_k^7 \pmod{41},$$

which is impossible if $k \equiv 0, 1, 2, 3, 4, 5, 6, 7 \pmod{10}$. This requires

$$k \not\equiv 0, 1, 2, 3, 4, 5, 6, 7 \pmod{10}.$$

Using (130), we get $z^2 \equiv 44 \pmod{61}$ if $k \equiv 9 \pmod{10}$. This gives a contradiction again, so $k \not\equiv 9 \pmod{10}$. From (132), we obtain

$$z^2 \equiv 7x_k^2 - y_k^7 \pmod{2521}.$$

Then $z^2 \equiv 1129 \pmod{2521}$ if $k \equiv 8 \pmod{10}$. But, $\left(\frac{1129}{2521}\right) = -1$. Hence, $k \not\equiv 8 \pmod{10}$.

Finally, we consider (107). Using (127), equation (123) implies that

$$z^2 \equiv 2x_k^7 + 11y_k^2 \pmod{23},$$

which is impossible if $k \equiv 0, 1, 2, 3, 4, 5, 6, 7 \pmod{10}$. This forces

$$k \not\equiv 0, 1, 2, 3, 4, 5, 6, 7 \pmod{10}.$$

It remains to consider $k \equiv 8, 9 \pmod{10}$. Here, we use equation (129). If $k \equiv 8 \pmod{10}$ or $k \equiv 9 \pmod{10}$, then $z^2 \equiv 30 \pmod{41}$ or $z^2 \equiv 35 \pmod{41}$, respectively. But, 30 and 35 are quadratic nonresidues modulo 41. So $k \not\equiv 8, 9 \pmod{10}$. Thus, the simultaneous Diophantine equations (122) and (123) can not be solved simultaneously. Hence, Theorem 3.6 is proved. □

PROOF OF THEOREM 3.7. We prove this theorem by showing the simultaneous Diophantine equations in (97) have no common solutions. Firstly, we consider the system of Diophantine equations

$$x^2 - 5y^2 = 4, \tag{133}$$

$$x^7 + 7y^2 = z^2, \tag{134}$$

where $x = L_n, y = F_n$ and $z = Z$. To prove this system has no solution, we follow the same approach used in the proof of Theorem 3.6 to exhaust all the possibilities of $k \equiv \rho \pmod{15}$ for $\rho = 0, 1, 2, \ldots, 14$, with using some equations of (110)-(121). Firstly, we consider (102). From (112), equation (134) gives

$$z^2 \equiv x_k^7 + 7y_k^2 \pmod{11}.$$

If $k \equiv 3 \pmod{15}$, then $z^2 \equiv 7 \pmod{11}$. This is impossible, so $k \not\equiv 3 \pmod{15}$. Using (117), we get $z^2 \equiv 51 \pmod{107}$ if $k \equiv 7 \pmod{15}$. But, $\left(\frac{51}{107}\right) = -1$. So $k \not\equiv 7 \pmod{15}$. From (118), we get a contradiction if $k \equiv 0, 1, 2, 5, 6, 8, 10, 12 \pmod{15}$. To exclude the rest possibilities, we use (119), which leads to

$$z^2 \equiv x_k^7 + 7y_k^2 \pmod{541}.$$

If $k \equiv 4 \pmod{15}$, then $z^2 \equiv 206 \pmod{541}$. This gives a contradiction since 206 is a quadratic nonresidue modulo 541. Similarly, $k \equiv 9, 11, 13, 14 \pmod{15}$ leads to a contradiction again. Therefore,

$$k \not\equiv 4, 9, 11, 13, 14 \pmod{15}.$$

Now, we consider (103). From (116), equation (134) implies that

$$z^2 \equiv 7y_k^2 - x_k^7 \pmod{61}.$$

Starting with $k \equiv 8 \pmod{15}$, we get $z^2 \equiv 43 \pmod{61}$. Again, we get a contradiction, thus $k \not\equiv 8 \pmod{15}$. Using (118), we get

$$z^2 \equiv x_k^7 + 7y_k^2 \pmod{181}.$$

If $k \equiv 0 \pmod{15}$, then $x_k \equiv x_0 \equiv 3 \pmod{181}$ and $y_k \equiv y_0 \equiv 180 \pmod{181}$, which give $z^2 \equiv 22 \pmod{181}$. This is impossible since $\left(\frac{22}{181}\right) = -1$. Furthermore, $k \equiv 3, 5, 7, 9, 10, 13, 14 \pmod{15}$ yields a contradiction again. Therefore,

$$k \not\equiv 0, 3, 5, 7, 9, 10, 13, 14 \pmod{15}.$$

From (119), the equation

$$z^2 \equiv x_k^7 + 7y_k^2 \pmod{541}$$

is impossible if $k \equiv 1, 2, 4, 6, 11 \pmod{15}$. So

$$k \not\equiv 1, 2, 4, 6, 11 \pmod{15}.$$

Using (120), we get

$$z^2 \equiv 7y_k^2 - x_k^7 \pmod{109441}.$$

If $k \equiv 12 \pmod{15}$, then $z^2 \equiv 98563 \pmod{109441}$. This is impossible since 98563 is a quadratic nonresidue modulo 109441. Hence, $k \not\equiv 12 \pmod{15}$.

Finally, we consider (104). Equation (118) leads to

$$z^2 \equiv x_k^7 + 7y_k^2 \pmod{181},$$

which is impossible if

$$k \equiv 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14 \pmod{15}.$$

Therefore, they are all excluded. Hence, the simultaneous Diophantine equations (133) and (134) have no common solutions. We finish the proof of the theorem by proving the system of the Diophantine equations

$$x^2 - 5y^2 = -4, \tag{135}$$

$$x^7 + 7y^2 = z^2, \tag{136}$$

where $x = L_n, y = F_n$ and $z = Z$, can not be solved simultaneously. Again, we use some appropriate equations of (124)-(132) to exclude all the possibilities of $k \equiv \rho \pmod{10}$ for $\rho = 0, 1, 2, \ldots, 9$. First of all, we consider equation (105). From (126), equation (136) leads to $z^2 \equiv 8 \pmod{11}$, $z^2 \equiv 6 \pmod{11}$ or $z^2 \equiv 10 \pmod{11}$ if $k \equiv 0 \pmod{10}$, $k \equiv 3 \pmod{10}$ or $k \equiv 4 \pmod{10}$, respectively. However, 6, 8 and 10 are quadratic nonresidues modulo 11. So $k \not\equiv 0, 3, 4 \pmod{10}$. Using (127), we get

$$z^2 \equiv 17x_k^2 + 11y_k^7 \pmod{23}.$$

If $k \equiv 1 \pmod{10}$, then $z^2 \equiv 21 \pmod{23}$. This yields a contradiction, hence $k \not\equiv 1 \pmod{10}$. If we use (129), we obtain

$$z^2 \equiv 7y_k^2 - x_k^7 \pmod{41},$$

which can not be held for $k \equiv 5, 6, 7, 8, 9 \pmod{10}$. Thus,

$$k \not\equiv 5, 6, 7, 8, 9 \pmod{10}.$$

From (130), we get $z^2 \equiv 31 \pmod{61}$ for $k \equiv 2 \pmod{10}$. But, $\left(\frac{31}{61}\right) = -1$. Therefore, $k \not\equiv 2 \pmod{10}$.

Next, we consider (106). In fact, equation (126) and $k \equiv 0 \pmod{10}$ give $z^2 \equiv 6 \pmod{11}$, again yielding a contradiction. So $k \not\equiv 0 \pmod{10}$. Similarly, we get a contradiction again if we use (129) for $k \equiv 1, 2, 3, 4, 5, 7 \pmod{10}$. Hence,

$$k \not\equiv 1, 2, 3, 4, 5, 7 \pmod{10}$$

From (131), we obtain

$$z^2 \equiv 95x_k^2 + 220y_k^7 \pmod{241}.$$

If $k \equiv 6 \pmod{10}$, then $z^2 \equiv 7 \pmod{241}$. Moreover, if $k \equiv 8 \pmod{10}$ or $k \equiv 9 \pmod{10}$, then $z^2 \equiv 21 \pmod{241}$ or $z^2 \equiv 37 \pmod{241}$, respectively. But, 7, 21 and 37 are quadratic nonresidues modulo 241. So $k \not\equiv 6, 8, 9 \pmod{10}$.

Lastly, we consider (107). From (126), we have $z^2 \equiv 8 \pmod{11}$ if $k \equiv 3 \pmod{10}$. This is impossible. Therefore, $k \not\equiv 3 \pmod{10}$. Equation (130) leads to a contradiction again if $k \equiv 6 \pmod{10}$. So $k \not\equiv 6 \pmod{10}$. In a similar way, we can use (129) to eliminate all the remaining possibilities of $k \equiv \rho \pmod{10}$ such that $\rho = 0, 1, 2, 4, 5, 7, 8, 9$. Hence, the simultaneous Diophantine equations (135) and (136) have no common solutions. Therefore, Theorem 3.7 is completely proved. $\square$

3.2. Solutions of generalizations of Markoff equation from linear recurrence sequences

From Subsection 1.2.2 of Chapter 1, we recall Markoff equation; that is, the Diophantine equation

$$x^2 + y^2 + z^2 = 3xyz \tag{137}$$

in positive integers $x \le y \le z$, which was deeply studied by Markoff [**160, 161**] demonstrating a relationship between its integer solutions (so-called Markoff triples) and Diophantine approximation. As mentioned earlier that Markoff obtained many interesting results related to the Markoff triples such as there are infinitely many Markoff triples, which can be generated from the fundamental solution $(1,1,1)$ and the branching operation

$$\begin{tikzcd}[column sep=small] & (x,y,z) \arrow[dl, dash] \arrow[dr, dash] & \\ (x,z,3xz-y) & & (y,z,3yz-x). \end{tikzcd}$$

In fact, the set of ordered positive solutions can be organized in a tree called Markoff's tree. A component of some Markoff triple is called a Markoff number. The sequence of Markoff numbers is as follows

$$1, 2, 5, 13, 29, 34, 89, 169, 194, 233, 433, 610, 985, 1325, 1597, \ldots$$

(sequence $A002559$ [**221**]), which appear as coordinates of Markoff triples

$$\begin{aligned} &(1,1,1), (1,1,2), (1,2,5), (1,5,13), (2,5,29), (1,13,34), (1,34,89), \\ &(2,29,169), (5,13,194), (1,89,233), (5,29,433), (1,233,610), \\ &(2,169,985), (13,34,1325), \ldots . \end{aligned}$$

This equation has been generalized by several authors. For instance, Hurwitz [**122**] applied Markoff's descent technique to the equation

$$x_1^2 + x_2^2 + \ldots + x_n^2 = ax_1x_2\ldots x_n,$$

with a being a nonzero integer and $n \ge 3$. Mordell [**175**] studied the integer solutions of the equation

$$x^2 + y^2 + z^2 = axyz + b,$$

where a and b are integers with $a > 0$. Another generalization was considered by Rosenberger [**198**], which has the form

$$ax^2 + by^2 + cz^2 = dxyz. \tag{138}$$

This equation is often called the Markoff-Rosenberger equation. Rosenberger proved that if $a, b, c, d \in \mathbb{N}$ are integers such that $\gcd(a,b) = \gcd(a,c) = \gcd(b,c) = 1$ and $a, b, c | d$, then nontrivial solutions exist only if $(a,b,c,d) \in T$, where

$$T = \{(1,1,1,1), (1,1,1,3), (1,1,2,2), (1,1,2,4), (1,1,5,5), (1,2,3,6)\}.$$

The Markoff-Rosenberger equation was generalized by Jin and Schmidt [**123**] in which they determined the positive integer solutions of the equation

$$AX^2 + BY^2 + CZ^2 = DXYZ + 1. \tag{139}$$

Jin and Schmidt showed that equation (139) has a fundamental solution if and only if

$$(A,B,C,D) \in \{(2,2,3,6),(2,1,2,2),(7,2,14,14),(3,1,6,6),\\ (6,10,15,30),(5,1,5,5),(1,t,t,2t)\}, \text{with } t \in \mathbb{N}.$$

Respecting the authors of this generalization, we call equation (139) the Jin-Schmidt equation. Other generalizations and studies related to the Markoff equation can be found in [**6**], [**15**], [**94**], [**119**] and the references given there. One of the recent studies related to the Markoff equation (137) was introduced by Luca and Srinivasan [**155**] in which they proved that the only solution of Markoff equation with $x \le y \le z$ such that $(x,y,z) = (F_i,F_j,F_k)$ is given by the well-known identity related to the Fibonacci numbers

$$1 + F_{2n-1}^2 + F_{2n+1}^2 = 3F_{2n-1}F_{2n+1}.$$

Moreover, Kafle, Srinivasan and Togbé [**125**] determined all triples of Pell numbers $(x,y,z) = (P_i,P_j,P_k)$ satisfying Markoff equation. Here, there is an other identity given by

$$2^2 + P_{2m-1}^2 + P_{2m+1}^2 = 3 \cdot 2 \cdot P_{2m-1}P_{2m+1}.$$

Recently, Altassan and Luca [**5**] considered the Markoff-Rosenberger equation (138) with integer solutions (x,y,z), which are all members of a Lucas sequence whose characteristic equation has roots which are quadratic units.

In this section, we present our new results in the following two subsections in which we extend the result of Luca and Srinivasan by simplifying their strategy with having upper bounds for the minimum of the indices to provide a direct approach for investigating such special solutions of the Jin-Schmidt equation (139) and the Markoff-Rosenberger equation (138), respectively. In the first subsection we find all the triples of Fibonacci numbers satisfying the Jin-Schmidt equation (the terms of the Fibonacci sequence $\{F_n\}$ are given by (23)), and in the other one we consider the generalized Lucas number solutions of the Markoff-Rosenberger equation (the generalized Lucas numbers are presented by the terms of the Lucas sequences $\{U_n\}$ and $\{V_n\}$ that are given by (19) and (20), respectively). Then we apply the obtained results to completely resolve concrete equations, e.g. we determine solutions containing only balancing numbers and Jacobsthal numbers, respectively (the terms of the sequences of balancing numbers $(B_n)_{n\ge0}$ and Jacobsthal numbers $(J_n)_{n\ge0}$ are respectively given by (29) and (27)). Following the same strategy, Markoff-Rosenberger triples containing only Fibonacci numbers were determined by Tengely [**244**].

3.2.1. Solutions of the Jin-Schmidt equation in Fibonacci numbers.

Here, we investigate the solutions $(X, Y, Z) = (F_I, F_J, F_K)$ in positive integers of the Jin-Schmidt equation (139). In other words, we study the solutions of the following Diophantine equations in the sequence of Fibonacci numbers:

$$2X^2 + 2Y^2 + 3Z^2 = 6XYZ + 1, \tag{140}$$

$$2X^2 + Y^2 + 2Z^2 = 2XYZ + 1, \tag{141}$$

$$7X^2 + 2Y^2 + 14Z^2 = 14XYZ + 1, \tag{142}$$

$$3X^2 + Y^2 + 6Z^2 = 6XYZ + 1, \tag{143}$$

$$6X^2 + 10Y^2 + 15Z^2 = 30XYZ + 1, \tag{144}$$

$$5X^2 + Y^2 + 5Z^2 = 5XYZ + 1. \tag{145}$$

We remark that the same technique can be applied in case of $(A, B, C, D) = (1, t, t, 2t)$ for given values of t. One of the interesting motivations about the Jin-Schmidt equation is that the equation (140) appeared in connection with the description of the lower part of the approximation spectrum for quaternions. Moreover, it is connected with the description of approximation constants for complex numbers on the circle $\{z \in \mathbb{C} \,|\, |z| = \frac{1}{\sqrt{2}}\}$ with respect to integers in the field $\mathbb{Q}(\sqrt{-3})$. For more details about these connections, one can see e.g. [**123**] and the references given there.

Next, we introduce the procedure which we use to study the existence and nonexistence of such special solutions of the Jin-Schmidt equation (particularly, equations (140)-(145)). Since this procedure can be adapted to study the solutions of any equation of the form $ax^2 + by^2 + cz^2 = dxyz + e$ (for certain nonzero integer coefficients) from certain binary linear recurrence sequences, we call it the general investigative procedure.

3.2.1.1. *General investigative procedure.*

To start the procedure off, we first have to obtain all the possible distinct equations

$$ax^2 + by^2 + cz^2 = dxyz + 1 \tag{146}$$

of equation (139) by permuting the coefficients A, B and C for

$$(A, B, C, D) \in S = \{(2, 2, 3, 6), (2, 1, 2, 2), (7, 2, 14, 14), (3, 1, 6, 6), (6, 10, 15, 30), (5, 1, 5, 5)\}.$$

The following steps summarize the technique of investigating all the solutions $(x, y, z) = (F_i, F_j, F_k)$ with $2 \le i \le j \le k$ for every equation of the form (146) for a given tuple (a, b, c, d); that is,

$$aF_i^2 + bF_j^2 + cF_k^2 = dF_iF_jF_k + 1, \tag{147}$$

where $2 \le i \le j \le k$. Note that we assumed that $i \ge 2$ since $F_1 = F_2 = 1$.

- Determining an upper bound for i in equation (147). We first rewrite the equation in the form

$$cF_k - dF_iF_j = -\frac{aF_i^2 + bF_j^2}{F_k} + \frac{1}{F_k}. \tag{148}$$

Inserting the values of F_i, F_j and F_k in the left-hand side of equation (148) with using the Binet's Fibonacci numbers formula in (31) (that is $F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$, where $(\alpha, \beta) = \left(\frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}\right)$ for all $n \geq 0$ and α is called the golden ratio and $\beta = \frac{-1}{\alpha}$), we obtain that

$$\begin{aligned} \frac{c}{\sqrt{5}}\alpha^k - \frac{d}{5}\alpha^{i+j} = {} & -\frac{aF_i^2 + bF_j^2}{F_k} + \frac{1}{F_k} + \frac{c}{\sqrt{5}}\beta^k \\ & - \frac{d}{5}(\alpha^i\beta^j + \alpha^j\beta^i - \beta^{i+j}). \end{aligned} \tag{149}$$

Based on the inequalities for the n^{th} Fibonacci number in (35) (that is $\alpha^{n-2} \leq F_n \leq \alpha^{n-1}$ holds for all $n \geq 1$) and $2 \leq i \leq j \leq k$ (that is $1 \leq F_i \leq F_j \leq F_k$), we have that

$$\frac{aF_i^2 + bF_j^2}{F_k} \leq (a+b)\frac{F_j^2}{F_k} \leq (a+b)\alpha^{2j-k} \leq (a+b)\alpha^j, \tag{150}$$

$$\frac{1}{F_k} \leq 1 < \alpha^j, \tag{151}$$

$$\left|\frac{c}{\sqrt{5}}\beta^k\right| = \left|-\frac{c}{\sqrt{5}}\alpha^{-k}\right| \leq \frac{c}{\sqrt{5}}\alpha^{-j} \leq \frac{c}{5}\alpha^j, \tag{152}$$

$$\left|\frac{d}{5}(\alpha^i\beta^j + \alpha^j\beta^i - \beta^{i+j})\right| \leq \frac{d}{5}(2\alpha^j + 1) \leq \frac{3d}{5}\alpha^j. \tag{153}$$

Taking the absolute values to equation (149) and using the inequalities (150)-(153), we obtain that

$$\left|\frac{c}{\sqrt{5}}\alpha^k - \frac{d}{5}\alpha^{i+j}\right| < (1 + a + b + \frac{c+3d}{5})\alpha^j.$$

Multiplying across by $\frac{\sqrt{5}}{c\alpha^{i+j}}$, we get that

$$\left|\alpha^{k-i-j} - \frac{d}{c\sqrt{5}}\right| < \frac{h}{\alpha^i}, \tag{154}$$

where $h = \frac{\sqrt{5}}{c}(1 + a + b + \frac{c+3d}{5})$. Suppose that

$$\min_{n\in\mathbb{Z}}\left|\alpha^n - \frac{d}{c\sqrt{5}}\right| > g > 0,$$

so inequality (154) implies that

$$g < \frac{h}{\alpha^i},$$

which clearly gives

$$i \leq \left\lfloor \frac{\ln(\frac{h}{g})}{\ln(\alpha)} \right\rfloor = l. \tag{155}$$

- Determining an upper bound for $k - j$ in equation (147). In fact, for a given i one can use inequality (154) to obtain an upper bound for $k - j$. Here, we provide such a bound using the upper bound for i (that is $i \leq l$) and inequality (154). We have that $1 \leq a, b, c \leq 15$ and $2 \leq d = D \leq 30$, which imply that $h \leq 52\sqrt{5} < 116.3$. Therefore, inequality (154) becomes

$$\left| \left| \alpha^{k-i-j} \right| - \left| \frac{d}{c\sqrt{5}} \right| \right| \leq \left| \alpha^{k-i-j} - \frac{d}{c\sqrt{5}} \right| < \frac{116.3}{\alpha^2} < 44.5$$

as $i \geq 2$, which leads to

$$\left| \alpha^{k-i-j} \right| < 44.5 + \left| \frac{d}{c\sqrt{5}} \right| < 44.5 + \frac{30}{\sqrt{5}} < 58$$

as $d \leq 30$ and $c \geq 1$. Hence,

$$k - j < i + \frac{\ln(58)}{\ln(\alpha)} < l + 9 \quad \text{or} \quad k \leq j + l + 8 \tag{156}$$

as $i \leq l$.

- Eliminating the values of i for $i \in [2, l]$ in which equation (147) does not hold (and then equation (146) for which $(x, y, z) = (F_i, F_j, F_k)$ with $2 \leq i \leq j \leq k$). For that, we solve the Diophantine equation

$$aF_i^2 + by^2 + cz^2 - dF_i yz - 1 = 0 \tag{157}$$

for y and z. This can be done by SageMath [**231**] using the function `solve_diophantine()`. If there exists no i for which equation (157) is satisfied, then equation (146) does not have any solution $(x, y, z) = (F_i, F_j, F_k)$ with $2 \leq i \leq j \leq k$ at the tuple (a, b, c, d).

- Fixing i and k for an arbitrary $k \in \{j, j+1, \ldots, j+l+8\}$ in equation (147), we get that

$$bF_j^2 - sF_j + w = 0, \tag{158}$$

where $s = dF_iF_k$ and $w = aF_i^2 + cF_k^2 - 1$. We note that the equation above only depends on j for all $j \geq i \geq 2$.

- Determining whether there exists j for which equation (158) holds using any of the following arguments:

(i) The technique of using the quadratic formula and the identity relationship between the Fibonacci numbers and Lucas numbers. Indeed, the sequence of Lucas numbers $(L_k)_{k\geq 0}$ is defined by (25), and from (21) the terms of the Fibonacci and Lucas sequences satisfy the identity

$$L_k^2 = 5F_k^2 \pm 4. \tag{159}$$

Multiplying (158) by $4b$ and adding s^2 to both sides lead to

$$(2bF_j - s)^2 = s^2 - 4bw. \tag{160}$$

Multiplying equations (159) and (160) together yields

$$Y_1^2 = (5X_1^2 \pm 4)(d^2F_i^2X_1^2 - 4b(aF_i^2 + cX_1^2 - 1)),$$

where $X_1 = F_k$ and $Y_1 = L_k(2bF_j - dF_iF_k)$. Therefore, our problem is reduced to obtain integral points on these biquadratic genus 1 curves. This will be realized using an algorithm implemented in Magma [**33**] as `SIntegralLjunggrenPoints()` (based on results obtained by Tzanakis [**249**]) or an algorithm described by Alekseyev and Tengely [**4**] in which they gave an algorithmic reduction of the search for integral points on such a curve to solving a finite number of Thue equations.

(ii) The Fibonacci identities substitution technique in which we use the Fibonacci sequence formula or some related identities to eliminate equation (158).

(iii) The congruence argument technique in which we eliminate equation (158) modulo a prime number p.

Applications of these arguments will be shown in details in the proof of Theorem 3.8.

- From every obtained solution $(x,y,z) = (F_i, F_j, F_k)$ of equation (146) at the tuple (a,b,c,d), we derive the corresponding solution $(X,Y,Z) = (F_I, F_J, F_K)$ of equation (139) at the tuple (A,B,C,D) by comparing the positions of the components of their tuples.

3.2.1.2. *New result.*

In the following we present our main result regarding the solutions of the Jin-Schmidt equation (139) in Fibonacci numbers. This result appears as Theorem 3.1 in [**107**].

THEOREM 3.8. *Let m be a positive integer greater than* 1. *If $(X,Y,Z) = (F_I, F_J, F_K)$ is a solution of equation* (139) *with $(A,B,C,D) \in S$, then the complete list of solutions is given by*

Eq.	*(A,B,C,D)*	$\{(X,Y,Z)\}$
(140)	$(2,2,3,6)$	$\{(1,1,1),(1,2,1),(1,2,3),(2,1,1),(2,1,3),$ $(F_{2m-1},F_{2m+1},1),(F_{2m+1},F_{2m-1},1)\}$
(141)	$(2,1,2,2)$	$\{(2,3,2),(2,5,2),(2,5,8),(8,5,2)\}$
(142)	$(7,2,14,14)$	$\{(1,2,1),(1,5,1),(3,2,1),(3,2,5)\}$
(143)	$(3,1,6,6)$	$\{(1,2,1),(3,2,1),(3,2,5)\}$
(144)	$(6,10,15,30)$	$\{(1,1,1),(1,2,1),(1,2,3)\}$
(145)	$(5,1,5,5)$	$\{\}$

3.2.1.3. *Proof of the result.*

PROOF OF THEOREM 3.8. We follow the general investigative procedure described in 3.2.1.1 to obtain and prove the desired solutions in the theorem.

□ Case 1. If $(A,B,C,D)=(2,2,3,6)$. We start by permuting the coefficients of equation (140) to obtain the equations

$$2x^2+2y^2+3z^2=6xyz+1, \tag{161}$$

$$2x^2+3y^2+2z^2=6xyz+1, \tag{162}$$

$$3x^2+2y^2+2z^2=6xyz+1. \tag{163}$$

We investigate the solutions of these equations with the assumptions of $x=F_i, y=F_j$ and $z=F_k$ for $2\le i\le j\le k$. That will be pursued by determining upper bounds for i and $k-j$ as described in the procedure. Thus, we have that the upper bounds for i in equations (161),(162) and (163) are given by $i\le 8, i\le 7$ and $i\le 7$, respectively. Therefore, we get that $k-j\le 16, k-j\le 15$ and $k-j\le 15$ in case of equations (161),(162) and (163), respectively. Let us consider equation (161) with $(x,y,z)=(F_i,F_j,F_k)$; that is, $2F_i^2+2F_j^2+3F_k^2-6F_iF_jF_k-1=0$ with $2\le i\le j\le k$ such that $2\le i\le 8$ and $k\le j+16$. Here, we obtain that $i\in\{2,3,5,7\}$ in which the equation $2F_i^2+2y^2+3z^2-6F_iyz-1=0$ is solvable. For $i=2$ and $k\in\{j,j+1,\dots,j+16\}$, we have that

$$2F_j^2-6F_kF_j+3F_k^2+1=0. \tag{164}$$

For each k we determine the values of j in which the latter equation is satisfied; we consider the equation as a quadratic in F_j and follow the argument described in (i) with $(a,b,c,d)=(2,2,3,6)$. It remains to solve the quartic Diophantine equations

$$Y_1^2=60X_1^4+8X_1^2-32, \tag{165}$$

$$Y_1^2=60X_1^4-88X_1^2+32, \tag{166}$$

where $X_1=F_k$ and $Y_1=(4F_j-6F_k)L_k$. As mentioned earlier in the procedure that the integral points on these equations can be obtained using the Magma function `SIntegralLjunggrenPoints()`. For equation (165), we get that $X_1 \in \{\pm 1, \pm 3\}$. In case of equation (166), we have that $X_1 \in \{\pm 1\}$. It follows that $F_k = 1$ or 3. Hence,

- If $k = j$ and $F_k = 1$, then we have that $k = j = 2$. Therefore, we get the solution $(x,y,z) = (F_2, F_2, F_2) = (1,1,1)$.
- If $k \in [j+1, j+16]$, then $F_k = 1$ is impossible for all $j \geq 2$.
- If $k = j$ and $F_k = 3$, then we obtain that $k = j = 4$. However, the triple $(1,3,3)$ is clearly not a solution to equation (161).
- If $k = j+1$ and $F_k = 3$, then we get that $k = 4$ and $j = 3$. Thus, we have the solution $(x,y,z) = (1,2,3)$.
- If $k = j+2$ and $F_k = 3$, then we obtain that $k = 4$ and $j = 2$. Again, we have that $(x,y,z) = (1,1,3)$ is not a solution to equation (161).
- Finally, we have that $F_k = 3$ is impossible for all $k \in \{j+3, \ldots, j+16\}$ with $j \geq 2$.

Now, we deal with the case of $i = 3$ and $k \in [j, j+16]$ with $j \geq 3$. In a similar way, we follow the same argument on the following quadratic equation in F_j:

$$2F_j^2 - 12F_kF_j + 3F_k^2 + 7 = 0. \tag{167}$$

We obtain the equations

$$Y_1^2 = 600X_1^4 + 200X_1^2 - 224$$

and

$$Y_1^2 = 600X_1^4 - 760X_1^2 + 224,$$

with $X_1 = F_k$ and $Y_1 = (4F_j - 12F_k)L_k$. In the former equation, we get that $X_1 \in \{\pm 1, \pm 3\}$. In the latter equation, we obtain that $X_1 \in \{\pm 1\}$. It follows that $F_k = 1$ or 3, which leads to no solution of equation (161). Furthermore, using the same argument described in (i) one can show that equation (161) does not have any solution of the form $(x,y,z) = (F_i, F_j, F_k)$ with $i \leq j \leq k$ in the case of $i = 5$ or 7 and $k \in [j, j+16]$.

Next, for equation (162) we study the equation $2F_i^2 + 3F_j^2 + 2F_k^2 - 6F_iF_jF_k - 1 = 0$ with $2 \leq i \leq 7$ and $j \leq k \leq j+15$. Again, we get that $i \in \{2,3,5,7\}$ in which the equation $2F_i^2 + 3y^2 + 2z^2 - 6F_iyz - 1 = 0$ is solvable in y and z. Starting with $i = 2$ and $k \in \{j, j+1, \ldots, j+15\}$, and following the technique (i) we obtain that $F_k \in \{1, 2\}$. Therefore,

- If $k = j$ and $F_k = 1$, we get the solution $(x,y,z) = (1,1,1)$.
- If $j+1 \leq k \leq j+15$, then $F_k = 1$ is impossible for all $j \geq 2$.
- If $k = j$ and $F_k = 2$, we obtain that $(x,y,z) = (1,2,2)$, which is not a solution to equation (162).
- If $k = j+1$ and $F_k = 2$, we have the solution $(x,y,z) = (1,1,2)$.
- If $F_k = 2$ and $k \in \{j+2, \ldots, j+15\}$ for $j \geq 2$, then we get no solution to equation (162).

Using the same argument, one can prove that the remaining values of i do not lead to any solution of equation (162).

Finally, we deal with equation (163) in which we have that $2 \le i \le 7$ and $j \le k \le j+15$. Here, we get that the equation $3F_i^2 + 2y^2 + 2z^2 - 6F_i yz - 1 = 0$ is solvable only at $i \in \{2, 4\}$. If $i = 2$, then the equation $3F_i^2 + 2F_j^2 + 2F_k^2 - 6F_iF_jF_k - 1 = 0$ becomes

$$F_j^2 - 3F_kF_j + F_k^2 + 1 = 0, \tag{168}$$

where $k \in [j, j+15]$ and $j \ge 2$. We obtain the solutions of equation (168) (and then of equation (163) in case of $i = 2$) by direct substitutions of the values of k in the equation or using the arguments (ii) and (iii). Thus,

- If $k = j$, then equation (168) implies that $F_j = 1$ and we get the solution $(x, y, z) = (1, 1, 1)$.
- If $k = j + 1$, we claim that $(x, y, z) = (F_2, F_2, F_3) = (1, 1, 2)$ is the only solution to equation (163). To prove our claim, we must show that equation (168) does not hold at $k = j + 1$ for all $j \ge 3$. In other words, we follow the idea described in the argument (ii) by using the Fibonacci numbers formula (that is $F_j = F_{j-1} + F_{j-2}$ for $j > 2$) to show that $F_j^2 - 3F_jF_{j+1} + F_{j+1}^2 + 1 < 0$ for all $j \ge 3$. Hence, let us start with the left-hand side,

$$\begin{aligned} F_j^2 - 3F_jF_{j+1} + F_{j+1}^2 + 1 &= F_j^2 - 3F_j(F_{j-1} + F_j) \\ &\quad + (F_{j-1} + F_j)^2 + 1 \\ &= -F_j^2 - F_{j-1}F_j + F_{j-1}^2 + 1 \\ &= -F_j^2 - F_{j-2}F_{j-1} + 1 \\ &< 0 \qquad \text{for all } j \ge 3, \end{aligned} \tag{169}$$

  and this proves the claim.
- If $k = j + 2$, then equation (168) gives us the identity

$$F_j^2 - 3F_jF_{j+2} + F_{j+2}^2 + 1 = 0,$$

  which is valid for all $j = 2m - 1$ with $m \ge 2$. This identity was proven in 2018 by Hoare G. [**115**]. Therefore, we get the solution $(x, y, z) = (F_2, F_j, F_k) = (1, F_{2m-1}, F_{2m+1})$.
- If $k = j + 3$, then in a similar way of the technique described in (169) we can show that the equation $F_j^2 - 3F_jF_{j+3} + F_{j+3}^2 + 1 > 0$ for all $j \ge 2$. That is

$$\begin{aligned} F_j^2 - 3F_jF_{j+3} + F_{j+3}^2 + 1 &= F_jF_{j+1} + 2F_{j-1}F_j \\ &\quad + 3F_{j-1}F_{j+1} + F_{j-1}^2 + 1 \\ &> 0 \qquad \text{for all } j \ge 2. \end{aligned}$$

- Similarly, if $k = j + 4$ or $j + 5$, then we conclude the unsolvability of equation (168) by having that

$$\begin{aligned}F_j^2 - 3F_jF_{j+4} + F_{j+4}^2 + 1 &= 2F_{j-1}F_{j+2} + F_{j-1}F_{j+3} \\ &\quad + F_j^2 + F_{j+1}^2 + F_{j+3}^2 + 1 \\ &> 0 \qquad \text{for all } j \geq 2,\end{aligned}$$

or

$$\begin{aligned}F_j^2 - 3F_jF_{j+5} + F_{j+5}^2 + 1 &= F_j^2 + F_{j+3}^2 + F_{j+4}^2 + 3F_jF_{j+2} \\ &\quad + 4F_{j-1}F_{j+4} + 1 \\ &> 0 \qquad \text{for all } j \geq 2,\end{aligned}$$

respectively.
- If $k = j+6$, we have the equation, call it, $G_j = F_j^2 - 3F_jF_{j+6} + F_{j+6}^2 + 1 = 0$. Here, we may use a congruence argument to show the given equation does not hold for all $j \geq 2$ by finding a prime number p in which $G_j \not\equiv 0 \pmod p$. Let $p = 17$, then for all $j \geq 2$ we have that

$$\begin{aligned}G_j \pmod{17} \equiv\ & 6, 5, 7, 16, 5, 16, 7, 5, 6, 13, 14, 12, \\ & 3, 14, 3, 12, 14, 13, 6, 5, 7, \ldots,\end{aligned}$$

which clearly has period = 18. This contradicts that $G_{j\geq 2} = 0$. Hence, equation (163) has no solutions of the form $(x, y, z) = (F_i, F_j, F_k)$, where $i = 2, j \geq 2$ and $k = j + 6$.
- Similarly, if $k = j + 7$ and $p = 13$, then we get that

$$\begin{aligned}H_j \pmod{13} \equiv\ & 3, 9, 6, 12, 12, 1, 12, 12, 6, 9, 3, 3 \\ & 1, 3, 3, 9, 6, 12, 12, \ldots,\end{aligned}$$

where $H_j = F_j^2 - 3F_jF_{j+7} + F_{j+7}^2 + 1$ for all $j \geq 2$. Again, we get a contradiction. For the remaining values of k, one can show that equation (163) has no more solutions using either of the arguments described in (ii) or (iii). Finally, we deal with $i = 4$ that leads to $F_j^2 - 9F_jF_k + F_k^2 + 13 = 0$. Using the argument described in (i), we get that $F_k = 1$ or 2, which is impossible for all $k \in \{j, j+1, j+2, \ldots, j+15\}$ with $j \geq 4$.

Finally, we combine the solutions of equations (161),(162) and (163) and then permute their components in which they satisfy equation (140). Therefore, equation (140) has the following set of solutions:

$$\begin{aligned}&(X, Y, Z) = (F_I, F_J, F_K) \in \{(1, 1, 1), (1, 2, 1), (1, 2, 3), \\ &(2, 1, 1), (2, 1, 3), (F_{2m-1}, F_{2m+1}, 1), (F_{2m+1}, F_{2m-1}, 1)\},\end{aligned}$$

for all $m \geq 2$.

□ Case 2. If $(A, B, C, D) = (2, 1, 2, 2)$. Again, by permuting the coefficients of equation (141) we obtain the distinct equations

$$2x^2 + y^2 + 2z^2 = 2xyz + 1, \tag{170}$$

$$x^2 + 2y^2 + 2z^2 = 2xyz + 1, \tag{171}$$

$$2x^2 + 2y^2 + z^2 = 2xyz + 1. \tag{172}$$

To study the solutions of these equations for which $x = F_i, y = F_j$ and $z = F_k$ with $2 \le i \le j \le k$, we compute the upper bounds for i and $k - j$ using the inequalities (155) and (156), respectively. It follows that the upper bounds for i in equations (170), (171) and (172) are presented by $i \le 9, i \le 9$ and $i \le 10$, respectively. Therefore, we get that $k - j \le 17$ in equations (170) and (171) and $k - j \le 18$ in equation (172). Moreover, we have that the equations $2F_i^2 + y^2 + 2z^2 - 2F_iyz - 1 = 0$ with $2 \le i \le 9$ and $2F_i^2 + 2y^2 + z^2 - 2F_iyz - 1 = 0$ with $2 \le i \le 10$ are solvable only for $i = 3$ or 6. On the other hand, the equation $F_i^2 + 2y^2 + 2z^2 - 2F_iyz - 1 = 0$ with $2 \le i \le 9$ is solvable for $i \in \{2, 4, 5, 7\}$. As a result of using the argument described in (i), we obtain that the solutions of equation (170) and equation (172) are given by $(x, y, z) = (2, 5, 8)$ and $(x, y, z) \in \{(2, 2, 3), (2, 2, 5)\}$, respectively. On the other hand, we have no solution to equation (171). From these solutions, we conclude that the solutions of equation (141) are given by $(X, Y, Z) = (2, 3, 2), (2, 5, 2), (2, 5, 8)$ and $(8, 5, 2)$.

□ Case 3. If $(A, B, C, D) = (7, 2, 14, 14)$. In a similar way, from equation (142) we get the equations

$$7x^2 + 2y^2 + 14z^2 = 14xyz + 1, \tag{173}$$

$$2x^2 + 7y^2 + 14z^2 = 14xyz + 1, \tag{174}$$

$$7x^2 + 14y^2 + 2z^2 = 14xyz + 1, \tag{175}$$

$$14x^2 + 7y^2 + 2z^2 = 14xyz + 1, \tag{176}$$

$$14x^2 + 2y^2 + 7z^2 = 14xyz + 1, \tag{177}$$

$$2x^2 + 14y^2 + 7z^2 = 14xyz + 1. \tag{178}$$

Assuming that $x = F_i, y = F_j$ and $z = F_k$ with $2 \le i \le j \le k$, we get that $i \le 8$ and $i \le 9$ in which the equations (173)–(176) and equations (177)–(178) can be held, respectively. Hence, the upper bounds for $k - j$ in equations (173)–(176) and (177)–(178) are given by $k - j \le 16$ and $k - j \le 17$, respectively. Eliminating some of the values of i in the given ranges, we get that $i \in \{2, 4\}, i \in \{3, 5\}$, and $i \in \{2, 5\}$ in which equations; (173) and (175), (174) and (178), and (176) and (177) for which $x = F_i$ are solvable, respectively. As before, we use the argument described in (i) to get the solution $(x, y, z) = (2, 3, 5)$ to equation (174) and the solutions $(x, y, z) = (1, 1, 2)$ and $(1, 1, 5)$ to equations (175) and (176). Moreover, the equation (177) has no more solution other than $(x,y,z) = (1,2,3)$. The remaining equations have no such solutions. These solutions yield the solutions of (142), which are given by $(X,Y,Z) \in \{(1,2,1),(1,5,1),(3,2,1),(3,2,5)\}$.

□ `Case 4`. If $(A,B,C,D) = (3,1,6,6)$. By permuting the coefficients of equation (143), we have the following distinct equations

$$3x^2 + y^2 + 6z^2 = 6xyz + 1, \tag{179}$$

$$x^2 + 3y^2 + 6z^2 = 6xyz + 1, \tag{180}$$

$$x^2 + 6y^2 + 3z^2 = 6xyz + 1, \tag{181}$$

$$6x^2 + y^2 + 3z^2 = 6xyz + 1, \tag{182}$$

$$6x^2 + 3y^2 + z^2 = 6xyz + 1, \tag{183}$$

$$3x^2 + 6y^2 + z^2 = 6xyz + 1. \tag{184}$$

Suppose that $x = F_i, y = F_j$ and $z = F_k$ with $2 \le i \le j \le k$. As before, we obtain that $i \le 8$, $i \le 9$, and $i \le 12$ for which equations (179)–(180), (181)–(182), and (183)–(184) can be valid, respectively. The upper bounds of $k - j$ can be easily followed using inequality (156). Furthermore, we get that $i \in \{2,4\}, i \in \{2,3\}, i \in \{2,4,5\}$, and $i \in \{2,4,5,11\}$ in which the equations; (179) and (184), (180) and (181), (182), and (183) are solvable in y and z for all $x = F_i$, respectively. It remains to study the solutions of these equations using any of the arguments described in (i), (ii) or (iii), we obtain that the only solution for equation (180) is $(x,y,z) = (2,3,5)$ and for equation (182) is $(x,y,z) = (1,2,3)$. Moreover, equations (183) and (184) have no more solutions other than $(x,y,z) = (1,1,2)$. Finally, the remaining equations are unsolvable. Therefore, the solutions of the main equation (143) are given by $(X,Y,Z) = (1,2,1),(3,2,1)$ and $(3,2,5)$.

□ `Case 5`. If $(A,B,C,D) = (6,10,15,30)$. Permuting the coefficients of equation (144) gives the equations

$$6x^2 + 10y^2 + 15z^2 = 30xyz + 1, \tag{185}$$

$$10x^2 + 6y^2 + 15z^2 = 30xyz + 1, \tag{186}$$

$$6x^2 + 15y^2 + 10z^2 = 30xyz + 1, \tag{187}$$

$$15x^2 + 6y^2 + 10z^2 = 30xyz + 1, \tag{188}$$

$$10x^2 + 15y^2 + 6z^2 = 30xyz + 1, \tag{189}$$

$$15x^2 + 10y^2 + 6z^2 = 30xyz + 1. \tag{190}$$

Let $x = F_i, y = F_j$ and $z = F_k$ with $2 \le i \le j \le k$. Here, we get that $i \le 8$ and $i \le 7$ for which equations (185)–(186) and (187)–(190) can be solvable, respectively. Therefore, $k \le j + 16$ in equations (185)–(186) and $k \le j + 15$ in the equations (187)–(190). Eliminating some of these values of i, we obtain that $i \in \{2\}, i \in \{2,3\}$, and $i \in \{2,4\}$ for which equations; (185) and (187), (186) and (189), and (188) and (190) are satisfied for all $x = F_i$. Using any of the mentioned arguments mainly the one described in (i), we get that $(x,y,z) = (1,1,1)$ and $(1,2,3)$ are the solutions for equation (185) and $(x,y,z) = (1,1,1)$ is the only solution for equations (186), (189) and (190). The solutions of equations (187) and (188) are given by $(x,y,z) = (1,1,1)$ and $(1,1,2)$. Combining these solutions back to equation (144), we get that $(X,Y,Z) \in \{(1,1,1),(1,2,1),(1,2,3)\}$.

□ `Case` 6. If $(A,B,C,D) = (5,1,5,5)$. In a similar way, we investigate the solutions of the equations

$$5x^2 + y^2 + 5z^2 = 5xyz + 1, \tag{191}$$

$$x^2 + 5y^2 + 5z^2 = 5xyz + 1, \tag{192}$$

$$5x^2 + 5y^2 + z^2 = 5xyz + 1, \tag{193}$$

where $x = F_i, y = F_j$ and $z = F_k$ with $2 \leq i \leq j \leq k$. The upper bounds for i in which the given equations can be held are given by $i \leq 8$ and $i \leq 9$ in equations (191)–(192) and (193), respectively. These imply that $k - j \leq 16$ and $k-j \leq 17$ in equations (191)–(192) and (193), respectively. Furthermore, we get that $i \in \{2,3\}$ and $i \in \{2\}$ for which equations; (191) and (193), and (192) with $x = F_i$ are solvable in y and z, respectively. Using any of the arguments described in (i), (ii) or (iii) leads to the unsolvability of equations (191)–(193) for which $(x,y,z) = (F_i, F_j, F_k)$. Therefore, equation (145) has no solution of the form $(X,Y,Z) = (F_I, F_J, F_K)$ in positive integers.

Hence, Theorem 3.8 is completely proved. □

3.2.2. Solutions of the Markoff-Rosenberger equation in generalized Lucas numbers.

Here, we generalize the strategy presented in Subsection 3.2.1 by considering the generalized Lucas number solutions of the Markoff-Rosenberger equation (138). Indeed, we adopt the general investigative procedure presented in 3.2.1.1 in a more extended way to provide general results for the solutions $(x,y,z) = (R_i, R_j, R_k)$ of the Markoff-Rosenberger equation, where R_i denotes the i^{th} generalized Lucas number of first/second kind, i.e. $R_i = U_i$ or V_i. Then we apply the strategy of achieving these results to completely resolve concrete equations, e.g. we determine solutions containing only balancing numbers and Jacobsthal numbers, respectively.

Next, we mention some auxiliary results, which we need later to present and proof our main results.

3.2.2.1. *Auxiliary results.*

From (19) and (20), we respectively recall the Lucas sequences $\{U_n\}_{n\geq 0}$,$\{V_n\}_{n\geq 0}$ as follows

$$\begin{aligned} &U_0(P,Q) = 0, U_1(P,Q) = 1, \qquad && U_n(P,Q) = PU_{n-1}(P,Q) - QU_{n-2}(P,Q),\\ &V_0(P,Q) = 2, V_1(P,Q) = P, && V_n(P,Q) = PV_{n-1}(P,Q) - QV_{n-2}(P,Q), \end{aligned}$$

where neither P nor Q is zero.

REMARK 3.9. *Assume that $P^\star = -P$ and define*

$$U_0^\star = 0,\ U_1^\star = 1\ U_n^\star = P^\star U_{n-1}^\star - QU_{n-2}^\star,$$

$$V_0^\star = 2,\ V_1^\star = P^\star,\ V_n^\star = P^\star V_{n-1}^\star - QV_{n-2}^\star.$$

Then we have

$$U_n^\star = (-1)^{n+1}U_n, \quad V_n^\star = (-1)^n V_n.$$

Based on the above identities, here we only deal with sequences satisfying $P > 0$.

Furthermore, we assume that $0 < D = P^2 - 4Q$, $P \geq 2$ and $-P - 1 \leq Q \leq P - 1$. We exclude the cases with $P = 1$ to make the presentation simpler. However, if $P = 1$, then $-2 \leq Q \leq 0$. Therefore, there are only two sequences to be considered. Namely, the Fibonacci sequence with $(P,Q) = (1,-1)$ and the Jacobsthal sequence with $(P,Q) = (1,-2)$. The former one was completely solved in [**244**], and the latter one will be handled separately as an application presented in 3.2.2.3.2. The characteristic polynomial associated to the above sequences is given by $x^2 - Px + Q$. The roots of the characteristic polynomial can be written in the form

$$\alpha = \frac{P + \sqrt{D}}{2}, \quad \beta = \frac{P - \sqrt{D}}{2}$$

and we have $\alpha - \beta = \sqrt{D}, \alpha + \beta = P$ and $\alpha\beta = Q$. We note that the conditions $P \geq 2, D > 0$ and $-P - 1 \leq Q \leq P - 1$ imply that $\alpha \geq 2$ and $|\beta| \leq 1$. First, we justify the second statement, and then the first one. Since $P \geq 2$ and $-P - 1 \leq Q \leq P - 1$, we have

$$(P-2)^2 \leq P^2 - 4Q \leq (P+2)^2.$$

Therefore, $P - 2 \leq \sqrt{D} \leq P + 2$. We have that $\beta = \frac{P-\sqrt{D}}{2}$. Hence,

$$-1 \leq \beta \leq 1.$$

This implies that $\alpha \geq 2$. Indeed, if $P \geq 3$, then $\alpha = P - \beta \geq P - 1 \geq 2$. On the other hand, if $P = 2$, then $Q \in \{-3, -2, -1, 1\}$. The case $Q = 1$ is not convenient since it leads to the characteristic equation $x^2 - 2x + 1 = (x-1)^2$ which has a double root so $D = 0$. Thus, $Q \leq -1$, so

$$\alpha = (2 + \sqrt{4 - 4Q})/2 \geq (2 + \sqrt{8})/2 = 1 + \sqrt{2} > 2.$$

Also, $\alpha > |\beta|$. All this has relevance later. By Binet's formulas given by (22), we have that

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad V_n = \alpha^n + \beta^n.$$

We assume that

$$\alpha^{k-2} \leq \quad U_k \quad \leq 2\alpha^k, \tag{194}$$

$$2\alpha^{k-1} \leq \quad V_k \quad \leq 2\alpha^k \quad \text{for } k \geq 1, \tag{195}$$

and these will be fulfilled in case of $D > 0, P \geq 2$ and $-P-1 \leq Q \leq P-1$, $Q \neq 0$. The bounds on inequalities (194) and (195) are obtained as follows. The upper bounds are clear since $\alpha - \beta = \sqrt{D} \geq 1$, so

$$U_k = \frac{\alpha^k - \beta^k}{\alpha - \beta} \leq \alpha^k - \beta^k \leq 2\alpha^k \quad \text{for } k \geq 1.$$

Similarly, we get that $V_k = \alpha^k + \beta^k \leq 2\alpha^k$. For the lower bounds, first we note that if $\beta > 0$, or $\beta < 0$ and k is even, then

$$V_k = \alpha^k + \beta^k > \alpha^k \geq 2\alpha^{k-1}.$$

If $\beta < 0$ and k is odd, then

$$V_k = \alpha^k - |\beta|^k = (\alpha - |\beta|)(\alpha^{k-1} + \ldots + |\beta|^{k-1}) \geq 2\alpha^{k-1},$$

since $\alpha - |\beta| = \alpha + \beta = P \geq 2$. For U_k, we use a similar argument. If $\beta > 0$, then

$$U_k = \frac{\alpha^k - \beta^k}{\alpha - \beta} = \alpha^{k-1} + \alpha^{k-2}\beta + \ldots + \beta^{k-1} > \alpha^{k-1} > \alpha^{k-2}.$$

If $\beta < 0$, then

$$U_k \geq \frac{\alpha^k - |\beta|^k}{\alpha + |\beta|} > \frac{\alpha - |\beta|}{2\alpha}(\alpha^{k-1} + \ldots + |\beta|^{k-1}) \geq \alpha^{k-2},$$

where we used again the fact that $\alpha - |\beta| = \alpha + \beta = P \geq 2$. In general, we assume that if $\{R_n\}_{n\geq 0}$ is a nondegenerate Lucas sequence of the first or second kind, then there exist constants s_1, s_2, i_1, i_2 such that

$$s_1\alpha^{n-i_1} \leq \quad R_n \quad \leq s_2\alpha^{n+i_2} \quad \text{for } n \geq 1,$$

and this will be fulfilled in the cases that we investigate later.

REMARK 3.10. *Let $\{R_n\}_{n\geq 0}$ be a binary linear recurrence sequence represented by $\{U_n\}_{n\geq 0}$ or $\{V_n\}_{n\geq 0}$. In order to determine all triples (R_i, R_j, R_k) satisfying equation* (138) *at a given tuple $(a, b, c, d) \in T$, where*

$$T = \{(1,1,1,1), (1,1,1,3), (1,1,2,2), (1,1,2,4), (1,1,5,5), (1,2,3,6)\}.$$

We first compute an upper bound for i (such that $1 \leq i \leq j \leq k$), denote it by $\mathtt{ub}_{R_n}(a, b, c, d)$. Hence, to resolve the equation completely with e.g. $(a, b, c, d) = (1, 2, 3, 6)$ one needs to handle the cases with $i \leq \mathtt{ub}_{R_n}(1, 2, 3, 6), i \leq \mathtt{ub}_{R_n}(1, 3, 2, 6), i \leq \mathtt{ub}_{R_n}(2, 1, 3, 6), i \leq \mathtt{ub}_{R_n}(2, 3, 1, 6), i \leq \mathtt{ub}_{R_n}(3, 1, 2, 6)$ and

$i \le \mathfrak{ub}_{R_n}(3,2,1,6)$. *Then after obtaining the solutions of* (138) *with these cases, we permute the components of these solutions in which they satisfy equation* (138) *at the tuple* $(a,b,c,d) = (1,2,3,6)$ *to determine the complete list of its solutions. Although, Theorem 3.11 gives the least upper bound for all such cases of the tuples of* T. *For that, we let* $\mathfrak{T}$ *be the set of all distinct tuples* (a,b,c,d) *derived from permuting the first three components of elements in* T.

3.2.2.2. *New results.*

Here, we present our main results regarding the generalized Lucas number solutions of the Markoff-Rosenberger equation (138). These results will appear in [**110**].

THEOREM 3.11. *Let* $(a,b,c,d) \in \mathfrak{T}$, $P \ge 2, -P-1 \le Q \le P-1$ *such that* $Q \ne 0$, $D > 0$ *and*

$$B_0 = \min_{I \in \mathbb{Z}} \left| \alpha^I - \frac{d}{c\sqrt{D}} \right|, \quad B_1 = \min_{I \in \mathbb{Z}} \left| \alpha^I - \frac{d}{c} \right|.$$

If $B_0 \ne 0$, *then* $B_0 \ge \alpha^{-4}$ *and if* $B_1 \ne 0$, *then* $B_1 \ge 0.17$. *Furthermore, if* $x = U_i, y = U_j$ *and* $z = U_k$ *with* $1 \le i \le j \le k$ *is a solution of* (138) *and* $B_0 \ne 0$, *then* $i \le 12$. *If* $x = V_i, y = V_j$ *and* $z = V_k$ *with* $1 \le i \le j \le k$ *is a solution of* (138) *and* $B_1 \ne 0$, *then* $i \le 7$.

PROOF. Let us start proving the first part of the theorem in which we show that $B_0 \ge \alpha^{-4}$ and $B_1 \ge 0.26$ as $B_0 \ne 0$ and $B_1 \ne 0$, respectively. We start with the case of B_1. From $\mathfrak{T}$, we have the rational number d/c is in the set $\{1,2,3,4,5,6\}$. If $I = 0$ and $B_1 \ne 0$, then $d/c \in \{2,3,4,5,6\}$. So $B_1 \ge 1$. However, if $d/c = 1$, then $B_1 = 0$ is achieved at $I = 0$ independently on P and Q. If $I < 0$, then $\alpha^I \le \alpha^{-1} \le 1/2$, which implies that $B_1 \ge 1/2$. Next, assume that $I = 1$. If $d/c = 1$, then $B_1 \ge 1$ since $\alpha \ge 2$. But, $B_1 = 0$ in case of $\alpha = d/c \in \{2,3,4,5,6\}$. Now, we indicate the values of P and Q in each of these cases giving that $B_1 = 0$. Since α and P are positive integers such that $\alpha \in \{2,3,4,5,6\}$ and $P \ge 2$. Then $\beta = P - \alpha$ must be an integer. Thus, we obtain that $\beta \in \{-1,1\}$ since $-1 \le \beta \le 1$. Furthermore, we get that $D = (\alpha-\beta)^2 \ge 1$ as $\alpha \ge 2$ and $\beta = \pm 1$. Therefore, the appropriate values of P and Q can be determined by

$$P = \alpha + \beta, \quad Q = \alpha\beta,$$

where $\alpha \in \{2,3,4,5,6\}$ and $\beta \in \{-1,1\}$. In the following table, we summarize the details of computations for the values of P and Q (such that $P \ge 2$ and $-P-1 \le Q \le P-1, Q \ne 0$) in which we have $B_1 = 0$.

α	(P,Q)
2	$(3,2)$
3	$(4,3),(2,-3)$
4	$(5,4),(3,-4)$
5	$(6,5),(4,-5)$
6	$(7,6),(5,-6)$

Assuming that $I \geq 2$. If $P \geq 4$, then $\alpha^I \geq \alpha^2 \geq (P-1)^2 \geq 9$. So $B_1 \geq 3$. Thus, it remains to deal with the cases $P \in \{2,3\}$ and $-P-1 \leq Q \leq P-1$ such that $Q \neq 0$. We start with $P = 2$ and $-3 \leq Q \leq 1$. As mentioned earlier the case with $Q = 1$ is not convenient since $D = 0$. So $Q \in \{-3,-2,-1\}$. If $P = 2$ and $Q = -3$, then $\alpha = (P + \sqrt{P^2 - 4Q})/2 = (2 + \sqrt{16})/2 = 3$. This gives us $\alpha^I \geq 9$. Thus, $B_1 \geq 3$. Next, if $P = 2$ and $Q = -2$, then $\alpha = 1 + \sqrt{3}$ and $\alpha^I \geq 2(2+\sqrt{3})$, which give $B_1 > 1.46$. Similarly, in case of $P = 2$ and $Q = -1$ we get $\alpha^I \geq 3 + 2\sqrt{2}$. Therefore, $B_1 \geq 3 - 2\sqrt{2} \simeq 0.17$. In the following table, we provide details of computations for the remaining cases in which we have B_1 is nonzero for all $I \geq 2$.

(P,Q)	α	lower bound on B_1
$(3,-4)$	4	$B_1 \geq 10$
$(3,-3)$	$(3+\sqrt{21})/2$	$B_1 > 8.37$
$(3,-2)$	$(3+\sqrt{17})/2$	$B_1 > 6.68$
$(3,-1)$	$(3+\sqrt{13})/2$	$B_1 > 4.90$
$(3,1)$	$(3+\sqrt{5})/2$	$B_1 > 0.85$

Indeed, the only special case in which we have $B_1 = 0$ is with $(P,Q) = (3,2)$. Here, $B_1 = 0$ is achieved at $I = 2$ and $d/c = 4$ since $\alpha = 2$. However, if $I \geq 3$, then $\alpha^I \geq 9$. So $B_1 \geq 3$. The computations above show that $B_1 \geq 0.17$.

We now turn to B_0. We have that $\sqrt{D} = \alpha - \beta \in [\alpha - 1, \alpha + 1]$. If $I \leq -2$, then

$$B_0 \geq \frac{(d/c)}{\sqrt{D}} - \frac{1}{\alpha^2} \geq \frac{1}{\alpha+1} - \frac{1}{\alpha^2} = \frac{\alpha^2 - \alpha - 1}{\alpha^2(\alpha+1)} > \frac{1}{\alpha^4},$$

since $\alpha \geq 2$, so $\alpha^2 - \alpha - 1 \geq 1$. If $I = -1$, then either $d/c \geq 2$, so

$$B_0 \geq \frac{2}{\alpha - \beta} - \frac{1}{\alpha} \geq \frac{2}{\alpha+1} - \frac{1}{\alpha} = \frac{\alpha - 1}{\alpha(\alpha+1)} \geq \frac{1}{\alpha^3} > \frac{1}{\alpha^4},$$

or $d/c = 1$ so

$$B_0 = \left|\frac{1}{\alpha} - \frac{1}{\alpha - \beta}\right| = \frac{|\beta|}{\alpha(\alpha - \beta)} \geq \frac{1}{\alpha^2(\alpha+1)} \geq \frac{1}{\alpha^4},$$

where we used the fact that $|\beta| = |Q|/\alpha \geq 1/\alpha$. In particular, the expression under the minimum to compute B_0 is not zero when I is negative. Assume next that $I \geq 0$. If $\beta \in \{-1,1\}$, then $\sqrt{D} = \alpha - \beta$ is an integer. Thus, in this case when $B_0 \neq 0$, the number B_0 is a positive rational number of denominator $c(\alpha - \beta) \leq 5(\alpha + 1)$. Therefore,

$$B_0 \geq \frac{1}{5(\alpha+1)} \geq \frac{1}{\alpha^4},$$

where the last inequality holds since $\alpha \geq 2$. Finally, if $\beta \in (-1,1)$, then $\left|c\sqrt{D}\alpha^I - d\right|$ is a quadratic real algebraic integer multiple of c. Its conjugate is

$$\left|-c\sqrt{D}\beta^I - d\right| = \left|c(\alpha - \beta)\beta^I + d\right| \leq c\alpha + (c+d).$$

Hence, since the norm of a quadratic nonzero algebraic integer that is divisible by c is greater than or equal to c^2, we get that

$$B_0 = \frac{\left|c\sqrt{D}\alpha^I - d\right|}{c\sqrt{D}} \geq \frac{1}{c\sqrt{D}}\left(\frac{c^2}{(c\alpha + c + d)}\right)$$
$$= \frac{1}{(\alpha-\beta)(\alpha+1+d/c)} \geq \frac{1}{(\alpha+1)(\alpha+7)} \geq \frac{1}{\alpha^4}.$$

We finish the proof by justifying the last inequality. If $P = 2$, then $\alpha \geq 1 + \sqrt{2}$ and the last inequality holds. If $P \geq 4$, then $\alpha = P - \beta > 3$, so the last inequality holds. It also holds if $P = 3$ and $Q < 0$, since then again $\alpha > 3$. Finally, if $P = 3$ and $Q > 0$, then $Q = 1, 2$. The case $Q = 2$ gives $\beta = 1$, a case already treated, and if $Q = 1$, then α is the square of the golden section so it is greater than $1 + \sqrt{2}$ and the desired inequality holds anyway.

Now, we prove the second part of the theorem. In fact, we look for solutions satisfying $x = U_i, y = U_j$ and $z = U_k$ with $1 \leq i \leq j \leq k$. We have that

$$\frac{c\alpha^k}{\sqrt{D}} - \frac{d}{D}\alpha^{i+j} = -\frac{aU_i^2 + bU_j^2}{U_k} + \frac{c\beta^k}{\sqrt{D}} - \frac{d}{D}(\alpha^i\beta^j + \alpha^j\beta^i - \beta^{i+j}).$$

We apply (194) to get an upper bound for $\frac{aU_i^2+bU_j^2}{U_k}$ such that $aU_i^2 + bU_j^2 \leq (a+b)U_j^2$ holds since the Lucas sequence $\{U_n\}_{n\geq 0}$ is monotone increasing. We obtain that

$$\frac{aU_i^2 + bU_j^2}{U_k} \leq \frac{(a+b)U_j^2}{U_k} \leq 4(a+b)\alpha^2\alpha^j.$$

Since $|\beta| \leq 1$, we get that

$$\left|\frac{c\beta^k}{\sqrt{D}}\right| \leq \left|\frac{c}{\sqrt{D}}\right| \leq \left|\frac{c\alpha^j}{\sqrt{D}}\right|.$$

The last expression to bound is $\frac{d}{D}(\alpha^i\beta^j + \alpha^j\beta^i - \beta^{i+j})$. In this case, we obtain that

$$\left|\frac{d}{D}(\alpha^i\beta^j + \alpha^j\beta^i - \beta^{i+j})\right| \leq \frac{d}{D}(2\alpha^j + 1).$$

Hence, we have that

$$\left|\frac{d}{D}(\alpha^i\beta^j + \alpha^j\beta^i - \beta^{i+j})\right| \leq \frac{3d}{D}\alpha^j.$$

From the above inequalities, we get that

$$\left|\frac{c\alpha^k}{\sqrt{D}} - \frac{d}{D}\alpha^{i+j}\right| \leq \left(4(a+b)\alpha^2 + \frac{c}{\sqrt{D}} + \frac{3d}{D}\right)\alpha^j.$$

It follows that

$$\left|\alpha^{k-i-j} - \frac{d}{c\sqrt{D}}\right| \leq \left(4(a+b)\alpha^2\frac{\sqrt{D}}{c} + \frac{3d}{c\sqrt{D}} + 1\right)\alpha^{-i}. \tag{196}$$

Let

$$B_0 = \min_{I\in\mathbb{Z}} \left| \alpha^I - \frac{d}{c\sqrt{D}} \right|.$$

If $B_0 \neq 0$ (then $B_0 \geq \alpha^{-4}$), then we get an upper bound for i from the inequality

$$\alpha^i \leq \frac{1}{B_0}\left(4(a+b)\alpha^2\frac{\sqrt{D}}{c} + \frac{3d}{c\sqrt{D}} + 1\right). \tag{197}$$

Since $B_0 \geq \alpha^{-4}, a+b \leq 6, 1 \leq \sqrt{D} \leq \alpha+1, c \geq 1$ and $d/c \leq 6$, then (197) becomes

$$\alpha^i \leq \alpha^4(4\cdot 6\cdot \alpha^2(\alpha+1)+19) < \alpha^{13},$$

where the last inequality holds since $\alpha \geq 2$. Thus, $i \leq 12$. In a similar way, one can prove the second part of the statement. Hence, we note that we get the inequalities (assuming that $B_1 \neq 0$, then $B_1 \geq 0.17$)

$$\left|\alpha^{k-i-j} - \frac{d}{c}\right| \leq \left(2(a+b)\alpha\frac{1}{c} + \frac{3d}{c} + 1\right)\alpha^{-i}. \tag{198}$$

$$\alpha^i \leq \frac{1}{B_1}\left(2(a+b)\alpha\frac{1}{c} + \frac{3d}{c} + 1\right). \tag{199}$$

Again, since $B_1 \geq 0.17, a+b \leq 6, c \geq 1$ and $d/c \leq 6$, so (199) is

$$\alpha^i \leq (0.17)^{-1}(2\cdot 6\cdot\alpha + 19) < \alpha^8,$$

where the last inequality holds since $\alpha \geq 2$. So $i \leq 7$. Hence, Theorem 3.11 is completely proved. □

It is important to remark that these lower bounds on B_0 or B_1 (namely, $B_0 \geq \alpha^{-4}$ or $B_1 \geq 0.17$) are the greatest lower bounds in case of any Lucas sequence of the first or second kind with all the tuples $(a,b,c,d) \in \mathfrak{T}$, respectively. Indeed, they may be greater due to particular sequences with certain tuples $(a,b,c,d) \in \mathfrak{T}$. A similar idea goes for the upper bounds on $i's$ (i.e. $i \leq 12$ or $i \leq 7$), they are only least upper bounds in case of any Lucas sequence of the first or second kind with all the tuples $(a,b,c,d) \in \mathfrak{T}$, respectively. Indeed, they could be smaller due to particular sequences and tuples. Moreover, note that in the proof of Theorem 3.11, the cases where we have $B_1 = 0$ were completely studied. Thus, it remains to classify the cases satisfying $B_0 = 0$, the result is as follows.

PROPOSITION 3.12. *If $P \geq 2$, $-P-1 \leq Q \leq P-1$, $Q \neq 0$ and $D > 0$, then $B_0 \neq 0$ fulfills unless*

- $e = 1,\ P = 3, Q = 2,\ \alpha = 2,\ \sqrt{D} = 1,\ I = 0,$
- $e = 2,\ P = 3, Q = 2,\ \alpha = 2,\ \sqrt{D} = 1,\ I = 1,$
- $e = 2,\ P = 4, Q = 3,\ \alpha = 3,\ \sqrt{D} = 2,\ I = 0,$
- $e = 3,\ P = 5, Q = 4,\ \alpha = 4,\ \sqrt{D} = 3,\ I = 0,$
- $e = 4,\ P = 3, Q = 2,\ \alpha = 2,\ \sqrt{D} = 1,\ I = 2,$
- $e = 4,\ P = 6, Q = 5,\ \alpha = 5,\ \sqrt{D} = 4,\ I = 0,$
- $e = 4,\ P = 2, Q = -3,\ \alpha = 3,\ \sqrt{D} = 4,\ I = 0,$

- $e = 5,\ P = 7, Q = 6,\ \alpha = 6,\ \sqrt{D} = 5,\ I = 0,$
- $e = 5,\ P = 3, Q = -4,\ \alpha = 4,\ \sqrt{D} = 5,\ I = 0,$
- $e = 6,\ P = 4, Q = 3,\ \alpha = 3,\ \sqrt{D} = 2,\ I = 1,$
- $e = 6,\ P = 8, Q = 7,\ \alpha = 7,\ \sqrt{D} = 6,\ I = 0,$
- $e = 6,\ P = 4, Q = -5,\ \alpha = 5,\ \sqrt{D} = 6,\ I = 0,$

where $e = d/c$ *such that* $(a,b,c,d) \in \mathfrak{T}$.

PROOF. From the proof of Theorem 3.11 (particularly, if $B_0 = 0$, then $B_0 \geq \alpha^{-4}$), it follows that B_0 cannot be zero if I is negative. Therefore, $I \geq 0$. If $\beta \in (-1,1)$, then B_0 cannot be zero if $I = 0$ (since $\sqrt{D}$ is not rational). So $I \neq 0$ and $\alpha^I = e/\sqrt{D}$, where $e = d/c$. Conjugating and taking ratios, we get $(\alpha/\beta)^I = -1$, which is false. Thus, $\beta = \pm 1$ and $\sqrt{D} = \alpha - \beta = P - 2\beta = P \pm 2$. Since $I \geq 0$, then $P \pm 2$ divides $e \in \{1,2,3,4,5,6\}$. So $P \leq 8$ (indeed, $2 \leq P \leq 8$). Since $\sqrt{D} = P - 2\beta > 0$, then $3 \leq P \leq 8$ in case of $\beta = 1$. Hence, it remains to study the cases in which we have

$$B_0 = \min_{I \geq 0} \left| \alpha^I - \frac{e}{\sqrt{D}} \right| = \min_{I \geq 0} \left| (P - \beta)^I - \frac{e}{(P - 2\beta)} \right| = 0.$$

Since $(P - \beta)^I$ is a positive integer as $\beta \in \{-1,1\}, I \geq 0$ and $2 \leq P \leq 8$ (avoiding that $P = 2$ in case of $\beta = 1$), so $e/(P - 2\beta)$ must be also a positive integer. In fact, the latter condition (i.e. $e/(P - 2\beta) \in \mathbb{Z}^+$) is achieved only at the following cases:

- □ $e = 1$, and $P - 2\beta = 1$, i.e. $(P,\beta) = (3,1)$.
- □ $e = 2$, and $P - 2\beta = 1,2$, i.e. $(P,\beta) = (3,1),(4,1)$, respectively.
- □ $e = 3$, and $P - 2\beta = 1,3$, i.e. $(P,\beta) = (3,1),(5,1)$, respectively.
- □ $e = 4$, and $P - 2\beta = 1,2,4$, i.e. $(P,\beta) = (3,1),(4,1),\ (6,1)$ or $(2,-1)$, respectively.
- □ $e = 5$, and $P - 2\beta = 1,5$, i.e. $(P,\beta) = (3,1),\ (7,1)$ or $(3,-1)$, respectively.
- □ $e = 6$, and $P - 2\beta = 1,2,3,6$, i.e. $(P,\beta) = (3,1),(4,1),(5,1),\ (8,1)$ or $(4,-1)$, respectively.

Finally, by examining which of the above cases leads to $B_0 = 0$, we get the results as follows. From the first case, we get that $B_0 = 0$ at $P = 3, \beta = 1, I = 0,\ e = 1$, and these give that $\sqrt{D} = P - 2\beta = 1, \alpha = P - \beta = 2$ and $Q = \alpha\beta = 2$. Hence, the first statement of the proposition is achieved. Similarly, from the second case we obtain that $B_0 = 0$ at $(P,\beta,I,e) = (3,1,1,2)$ and $(4,1,0,2)$. The former tuple implies that $\sqrt{D} = 1, \alpha = 2$ and $Q = 2$. So the second statement of the proposition is also fulfilled. However, the third statement is accomplished similarly by the latter tuple. We also get $B_0 = 0$ only in case of $P = 5, \beta = 1, I = 0,\ e = 3$, that give $\sqrt{D} = 3, \alpha = 4$ and $Q = 4$. Hence, the fourth statement of the proposition is obtained. In a very similar way, the remaining statements of the proposition will be fulfilled from the last three cases above. This completes the proof of Proposition 3.12. □

3.2.2.3. *Applications.*

Let $(a,b,c,d) \in T$ and $\{R_n\}_{n\geq 0}$ be either $\{U_n\}_{n\geq 0}$ or $\{V_n\}_{n\geq 0}$. In order to apply the procedure described in Theorem 3.11 to resolve equation (138) in $x = R_i, y = R_j, z = R_k$, we firstly do the following steps (since in Theorem 3.11, $1 \leq i \leq j \leq k$ is assumed). The first step is permuting the first three components in (a,b,c,d). Then for each of the resulting tuples, we provide an upper bound for i as explained in Theorem 3.11. In fact, Theorem 3.11 gives the least upper bound for all such cases of the tuples of T. After that, we adopt the arguments described the general investigative procedure presented in 3.2.1.1 (in case of the Fibonacci number solutions of equation (139)) to determine the list of solutions. Finally, when the solutions of (138) with these cases are obtained we permute the components of these solutions in which they satisfy equation (138) at the tuple $(a,b,c,d) \in T$ in order to determine all of its solutions $(x,y,z) = (R_i, R_j, R_k)$. If we fix (a,b,c,d), i and $m = k - j$, then we need to study the equation

$$aR_i^2 + bR_j^2 + cR_{j+m}^2 - dR_iR_jR_{j+m} = 0,$$

where $R_n = U_n$ or V_n. We note that the equation above only depends on j. Now, we adopt the arguments given the general investigative procedure.

(I) We eliminate as many values of i as possible by checking the solvability of quadratic equations

$$aR_i^2 + by^2 + cz^2 - dR_iyz = 0.$$

(II) For fixed m, we eliminate equations $aR_i^2 + bR_j^2 + cR_{j+m}^2 - dR_iR_jR_{j+m} = 0$ modulo p, where p is a prime.

(III) We can also eliminate equations $aR_i^2 + bR_j^2 + cR_{j+m}^2 - dR_iR_jR_{j+m} = 0$ using related identities of second order linear recurrence sequences.

(IV) We consider the equation $aR_i^2 + bR_j^2 + cR_{j+m}^2 = dR_iR_jR_{j+m}$ as a quadratic in R_j. Its discriminant $d^2R_i^2R_{j+m}^2 - 4b(aR_i^2 + cR_{j+m}^2)$ must be a square. As given in (21), the terms of the sequences $\{U_n\}_{n\geq 0}$ and $\{V_n\}_{n\geq 0}$ satisfy the fundamental identity

$$V_n^2 - DU_n^2 = 4Q^n.$$

Therefore, in case of $Q = \pm 1$ we have the systems of equations

$$\begin{aligned} Y_1^2 &= DX^2 \pm 4, \\ Y_2^2 &= d^2R_i^2X^2 - 4b(aR_i^2 + cX^2), \end{aligned}$$

where $X = R_{j+m} = U_{j+m}$, $Y_1 = V_{j+m}$, $Y_2 = 2bR_j - dR_iU_{j+m}$; and

$$\begin{aligned} Y_1^2 &= DX^2 \mp 4D, \\ Y_2^2 &= d^2R_i^2X^2 - 4b(aR_i^2 + cX^2), \end{aligned}$$

where $X = R_{j+m} = V_{j+m}$, $Y_1 = DU_{j+m}$, $Y_2 = 2bR_j - dR_iV_{j+m}$. Multiplying these equations together, in general, yields quartic genus 1 curves. One may determine the integral points on these curves using the Magma [**33**] function (based on results obtained by Tzanakis [**249**]) `SIntegralLjunggrenPoints`. Indeed, it may happen that we get genus 0 curves.

Let us apply the procedure described in Theorem 3.11 with these arguments to determine the solutions of equation (138) in some second order linear recurrence sequences. The results of the following applications will appear in [**110**].

3.2.2.3.1. Balancing numbers and Markoff-Rosenberger equations.

The first definition of balancing numbers is essentially due to Finkelstein [**83**], although he called them numerical centers. In 1999, Behera and Panda [**17**] defined balancing numbers as follows. A positive integer n is called a balancing number if

$$1 + 2 + \ldots + (n-1) = (n+1) + (n+2) + \ldots + (n+k)$$

for some $k \in \mathbb{N}$. The sequence of balancing numbers is denoted by $\{B_n\}_{n\geq 0}$. This sequence can be defined in a recursive way as well, we have that $B_0 = 0, B_1 = 1$ and

$$B_n = 6B_{n-1} - B_{n-2}, \quad n \geq 2.$$

As we see this is the sequence $\{U_n(6,1)\}_{n\geq 0}$. So $P = 6, Q = 1$ and $D = 32$. We also have that

$$\alpha = 3 + 2\sqrt{2}, \quad \beta = 3 - 2\sqrt{2}.$$

We have the bounds

$$\alpha^{n-1} \leq B_n \leq \alpha^n \quad \text{for } n \geq 1, \tag{200}$$

which are specific to the sequence of balancing numbers. Since $Q = 1$, the numbers $X = B_n$ satisfy the Pellian equation $Y^2 = 8X^2 + 1$. We prove the following result:

THEOREM 3.13. *If $(x,y,z) = (B_i, B_j, B_k)$ is a solution of the equation*

$$ax^2 + by^2 + cz^2 = dxyz$$

and $(a,b,c,d) \in \{(1,1,1,1),(1,1,1,3),(1,1,2,2),(1,1,2,4),(1,1,5,5),(1,2,3,6)\}$, then there is at most one solution given by $x = y = z = B_1 = 1$.

PROOF. Note that here we can directly use the results given in Theorem 3.11 (i.e. $B_0 \geq \alpha^{-4}$ and $i \leq 12$), argument (I) and any of the arguments described in (II), (III) or (IV) to prove the theorem completely. But in practice, having a smaller upper bound on i and eliminating as many $i's$ as possible are very useful for determining the complete set of solutions, and as pointed earlier the upper bound "12" on i is only the least upper bound that could be even smaller due to particular sequences. Therefore, we follow the general strategy given in Theorem 3.11 to compute the best possible values for the lower bounds on $B_0's$ (i.e. greater values) and the upper bounds on $i's$ (i.e. smaller values) specific to the sequence of balancing numbers with the use of the inequalities given in (200). It turns out that in all the cases we have that $i \leq 5$. Moreover, by applying argument (I), many values of $i's$ can be eliminated by checking integral solutions of binary quadratic forms. Therefore, we skip the congruence arguments given by (II) and (III). We directly consider the genus 1 curves obtained from the system of equations

$$\begin{aligned} Y_1^2 &= 8X^2 + 1, \\ Y_2^2 &= d^2 B_i^2 X^2 - 4b(aB_i^2 + cX^2). \end{aligned}$$

In the following table, we provide details of the computations.

$[a,b,c,d]$	B_0	C_0	$[i]$	$[i, A'X^4 + B'X^2 + C', [X,Y]]$
$[1,1,1,1]$	0.0052038	5	$[2]$	$[2, 256X^4 - 1120X^2 - 144, []]$
$[1,1,1,3]$	0.3587572	2	$[1]$	$[1, 40X^4 - 27X^2 - 4, [[1,-3],[-1,-3]]]$
$[1,1,2,2]$	0.0052038	4	$[2]$	$[2, 1088X^4 - 1016X^2 - 144, []]$
$[1,2,1,2]$	0.1819805	3	$[2]$	$[2, 1088X^4 - 2168X^2 - 288, []]$
$[2,1,1,2]$	0.1819805	3	$[]$	$[]$
$[1,1,2,4]$	0.1819805	2	$[1]$	$[1, 64X^4 - 24X^2 - 4, [[1,6],[-1,6]]]$
$[1,2,1,4]$	0.2928932	3	$[1]$	$[1, 64X^4 - 56X^2 - 8, [[1,0],[-1,0]]]$
$[2,1,1,4]$	0.2928932	3	$[1]$	$[1, 96X^4 - 52X^2 - 8, [[1,-6],[-1,-6]]]$
$[1,1,5,5]$	0.0052038	4	$[1]$	$[1, 40X^4 - 27X^2 - 4, [[1,-3],[-1,-3]]]$
$[1,5,1,5]$	0.1161165	4	$[1]$	$[1, 40X^4 - 155X^2 - 20, [[2,0],[-2,0]]]$
$[5,1,1,5]$	0.1161165	4	$[1]$	$[1, 168X^4 - 139X^2 - 20, [[1,-3],[-1,-3]]]$
$[1,2,3,6]$	0.1819805	2	$[1]$	$[1, 96X^4 - 52X^2 - 8, [[1,-6],[-1,-6]]]$
$[1,3,2,6]$	0.3587572	2	$[1]$	$[1, 96X^4 - 84X^2 - 12, [[1,0],[-1,0]]]$
$[2,1,3,6]$	0.1819805	2	$[1]$	$[1, 192X^4 - 40X^2 - 8, [[1,12],[-1,12]]]$
$[2,3,1,6]$	0.0606601	4	$[1]$	$[1, 192X^4 - 168X^2 - 24, [[1,0],[-1,0]]]$
$[3,1,2,6]$	0.3587572	2	$[1]$	$[1, 224X^4 - 68X^2 - 12, [[1,-12],[-1,-12]]]$
$[3,2,1,6]$	0.0606601	4	$[1]$	$[1, 224X^4 - 164X^2 - 24, [[1,6],[-1,6]]]$

The first column gives the tuples $(a,b,c,d) \in \mathfrak{T}$, the second column represents approximated lower bounds on $B_0's$, the third column has upper bounds on $i's$ represented by C_0, in the fourth column we provide lists containing the remaining values of $i's$ not eliminated by argument (I), and in the last column we have lists containing i, the right hand side of the quartic polynomial $Y^2 = A'X^4 + B'X^2 + C'$ defining genus 1 curve and the integral solutions (the second coordinate is only up to sign, for us, only the first coordinate is interesting since that gives B_{j+m}). For example, in case of $(a,b,c,d) = (1,2,3,6)$ we have that $B_0 \approx 0.1819805$ and $C_0 = 2$. That is $i \leq 2$. Applying argument (I), we can eliminate $i = 2$. Hence, it remains to study the case with $i = 1$. Here, we get that the only integral solutions are the ones with $X = \pm 1$. Since $X = B_{j+m}$, the only possibility is $B_{j+m} = 1$. The last step is the solution of the quadratic equation

$$1^2 + 2 \cdot B_j^2 + 3 \cdot 1^2 = 6 \cdot 1 \cdot B_j \cdot 1.$$

It follows that B_j is either 1 or 2, but 2 is not a balancing number. Therefore, the only solution in this case is

$$(B_i, B_j, B_k) = (B_1, B_1, B_1) = (1,1,1).$$

□

3.2.2.3.2. Jacobsthal numbers and Markoff-Rosenberger equations.

If $(P,Q) = (1,-2)$, then we deal with a special sequence in which we have that $P < 2$; that is the sequence of Jacobsthal numbers $\{J_n\}_{n\geq 0} = \{U_n(1,-2)\}_{n\geq 0}$. Here, we have $J_0 = 0, J_1 = 1$ and

$$J_n = J_{n-1} + 2J_{n-2} \quad \text{if } n \geq 2.$$

It is also known that the next Jacobsthal number is also given by the recursion formula

$$J_{n+1} = 2J_n + (-1)^n.$$

We obtain that

$$D = 9, \quad \alpha = 2, \quad \beta = -1.$$

Therefore, the closed-form of J_n is given by

$$\frac{2^n - (-1)^n}{3}.$$

Based on the above closed-form equation, we may provide bounds for J_n, these are as follows

$$\frac{2^{n-1}}{3} \leq J_n \leq 2^{n-1}, \quad n \geq 1. \tag{201}$$

Similarly, these bounds are only specific to the general term J_n. Here, we prove the following statement:

THEOREM 3.14. *If* $(x,y,z) = (J_i, J_j, J_k)$ *is a solution of the equation*

$$aJ_i^2 + bJ_j^2 + cJ_k^2 = dJ_iJ_jJ_k \tag{202}$$

and $(a,b,c,d) \in \{(1,1,1,1),(1,1,1,3),(1,1,2,2),(1,1,2,4),(1,1,5,5),(1,2,3,6)\}$, *then the complete list of solutions is given by*

(a,b,c,d)	*solutions*
$(1,1,1,1)$	$\{(3,3,3)\}$
$(1,1,1,3)$	$\{(1,1,1)\}$
$(1,1,2,2)$	$\{\}$
$(1,1,2,4)$	$\{(1,1,1),(1,3,1),(1,3,5),(3,1,1),(3,1,5),(3,11,1),(11,3,1)\}$
$(1,1,5,5)$	$\{(1,3,1),(3,1,1)\}$
$(1,2,3,6)$	$\{(1,1,1),(5,1,1)\}$

PROOF. Since $P < 2$, we cannot directly use the results given in Theorem 3.11. But, since $\beta = -1, \alpha = 2$ and $\sqrt{D} = 3$, we may follow the steps of the proof of Theorem 3.11 with the use of the inequalities given in (201). Hence, we obtain that

$$B_0 = \min_{I \in \mathbb{Z}} \left| 2^I - \frac{d}{3c} \right|$$

and

$$\left| 2^{k-i-j} - \frac{d}{3c} \right| \leq \left(\frac{9(a+b)}{2c} + \frac{d}{c} + 1 \right) 2^{-i}. \tag{203}$$

In fact, in some cases we obtain that $B_0 = 0$.

The case $(a,b,c,d) = (1,1,1,1)$. Here, we obtain that $B_0 \approx 0.0833333$ and the bound for i is 7. Applying the argument given by (I), it turns out that all values can be eliminated except $i = 3$. If $i = 3$, then we compute the possible values of $k - j$ from inequality (203). We have that $k - j \in \{0,1,2,3\}$. If $k - j \in \{1,2\}$, then applying (II) with $p = 3$ works and in case of $k - j = 3$ we use $p = 11$ to show that there is no solution. The remaining case is related to $k - j = 0$. We obtain the equation

$$3^2 + J_j^2 + J_j^2 = 3J_jJ_j.$$

It follows that $J_j = J_k = 3$, so the solution is given by $(J_i, J_j, J_k) = (3,3,3)$.

The case $(a,b,c,d) = (1,1,1,3)$. In this case in (203), we have $|2^{k-i-j} - 1|$ and this expression is 0 if $k - i - j = 0$. Therefore, we need to study the equation

$$(2^i - (-1)^i)^2 + (2^j - (-1)^j)^2 + (2^{i+j} - (-1)^{i+j})^2 = \\ (2^i - (-1)^i) \cdot (2^j - (-1)^j) \cdot (2^{i+j} - (-1)^{i+j}).$$

By symmetry we may assume that $i \leq j$. The small solutions with $0 \leq i \leq j \leq 2$ can be enumerated easily. Since we consider solutions with $i, j > 0$, we omit $(i,j) = (0,0)$. The other solution is given by $(i,j) = (1,1)$. Hence, we get that $(J_i, J_j, J_k) = (1,1,1)$. If $i = 2$, then it follows that with modulo 7 there is no solution. If $i > 2$, then we work modulo 8 to show that no solution exists. If $k - i - j \neq 0$, then we obtain that $B_0 = 1$. As a consequence, we have that $i \in \{1,2\}$. We may exclude the cases $i = 1,2$ and $k - j = 2$ modulo 5. In a similar way, working modulo 7 we eliminate the cases $i = 1, k - j = 3$ and $i = 2, k - j = 3,4$. The remaining cases are given by $i \in \{1,2\}, k - j \in \{0,1\}$. If $i = 1,2, k - j = 0$, then it easily follows that $(1,1,1)$ is the only solution. If $i = 1,2, k - j = 1$, then the equation is

$$1 + J_j^2 + J_{j+1}^2 = 3J_jJ_{j+1}.$$

Since $J_{j+1} = 2J_j + (-1)^j$, the above equation can be written as

$$J_j^2 - (-1)^j J_j - 2 = 0.$$

Thus, the only possibilities are given by $J_j \in \{\pm 1, \pm 2\}$. Again, the only solution we get is $(1,1,1)$.

The case $(a,b,c,d) = (1,1,2,2)$. Here, we compute the bounds for i in the cases $(a,b,c,d) = (1,1,2,2),(1,2,1,2),(2,1,1,2)$. Simply argument (I) is enough to show that there exists no solution.

The case $(a,b,c,d) = (1,1,2,4)$. The bound for i is 4, and by (I) we can eliminate the case $i = 4$ when the order of the coefficients is $(1,1,2,4)$. Congruence arguments (modulo 3 or 7) work if $(i,k-j) \in \{(1,2),(1,3),(2,2),(2,3)\}$. The remaining cases are

$$(i,k-j) \in \{(1,0),(1,1),(2,0),(2,1),(3,0),(3,1),(3,2),(3,3)\}.$$

From $(i,k-j) = (1,0),(2,0),(3,0)$, we obtain the solutions (by solving quadratic equations) $(1,1,1)$ and $(3,1,1)$. If $(i,k-j) = (1,1),(2,1)$, then we get

$$J_j^2 + 4(-1)^j J_j + 3 = 0.$$

Hence, $J_j = 1$ or 3. So we obtain the solutions $(1,1,1)$, $(1,3,1)$ and $(1,3,5)$. In case of $(i,k-j) = (3,1)$, we obtain

$$15J_j^2 + 4(-1)^j J_j - 11 = 0.$$

Thus, we have the solution $(3,1,1)$. By applying the rule $J_{n+1} = 2J_n + (-1)^n$ two or three times, we can reduce the problems $(i,k-j) = (3,2),(3,3)$ to quadratic equations. The formulas are getting more involved, for example if $(i,k-j) = (3,2)$ we have

$$9 + J_j^2 + 2(4J_j + 2(-1)^j + (-1)^{j+1})^2 = 12J_j(4J_j + 2(-1)^j + (-1)^{j+1}).$$

In this case, we get that $J_j = 1$. In a very similar way, we handle the cases with the tuples $(1,2,1,4)$ and $(2,1,1,4)$.

The case $(a,b,c,d) = (1,1,5,5)$. Here, we need to deal with the tuples $(1,1,5,5),(1,5,1,5)$ and $(5,1,1,5)$. The bounds for i are given by 3, 6 and 6, respectively. Since the steps are similar as we have applied in the previous cases, we omit the details.

The case $(a,b,c,d) = (1,2,3,6)$. We only provide some data related to the computation. Let us start with the bounds:

tuple	bound for i	special case
$(1,2,3,6)$	4	-
$(1,3,2,6)$	2	$k-i-j=0$
$(2,1,3,6)$	4	-
$(2,3,1,6)$	2	$k-i-j=1$
$(3,1,2,6)$	2	$k-i-j=0$
$(3,2,1,6)$	2	$k-i-j=1$

As before, we apply the arguments given by (I) and (II) and the identity $J_{n+1} = 2J_n + (-1)^n$ to resolve all the possible cases. The only new case that has not appeared yet is $k-i-j=1$. If we take the tuple $(2,3,1,6)$, then we obtain

$$2J_i^2 + 3J_j^2 + J_{i+j+1}^2 - 6J_iJ_jJ_{i+j+1} = 0, \tag{204}$$

or

$$2(2^i - (-1)^i)^2 + 3(2^j - (-1)^j)^2 + (2^{i+j+1} - (-1)^{i+j+1})^2 = \\ 2(2^i - (-1)^i) \cdot (2^j - (-1)^j) \cdot (2^{i+j+1} - (-1)^{i+j+1}). \quad (205)$$

With respect to the values of i and j, we consider the following cases (assuming that $1 \le i \le j \le k$):

- If i and j are both even, i.e. $i = 2t$ and $j = 2r$ for all positive integers $t, r \ge 1$, then equation (205) becomes

$$E_1 = 2(4^t - 1)^2 + 3(4^r - 1)^2 + (2 \cdot 4^{t+r} + 1)^2 \\ - 2(4^t - 1) \cdot (4^r - 1) \cdot (2 \cdot 4^{t+r} + 1) = 0.$$

  But, $E_1 \pmod 8 \equiv 4$ for all $t, r \ge 1$, which leads to a contradiction. Moreover, since i and j are both even with $i \ge 2$ and $j \ge 2$, then all the even values of i and j are excluded.
- If i and j are both odd, i.e. $i = 2t + 1$ and $j = 2r + 1$ for all positive integers $t, r \ge 1$, then equation (205) implies that

$$E_2 = 2(2 \cdot 4^t + 1)^2 + 3(2 \cdot 4^r + 1)^2 + (2 \cdot 4^{t+r+1} + 1)^2 \\ - 2(2 \cdot 4^t + 1) \cdot (2 \cdot 4^r + 1) \cdot (2 \cdot 4^{t+r+1} + 1) = 0.$$

  Similarly, $E_2 \pmod 8 \equiv 4$ for all $t, r \ge 1$, and again we get a contradiction. Indeed, all the odd values of $i \ge 3$ and $j \ge 3$ are excluded, and it remains only to check whether equation (204) has solutions or not at the following cases: $i = 1, j = 1; i = 1, j \ge 3; j = 1, i \ge 3$. In fact, since we assumed that $1 \le i \le j \le k$, then the latter case can be covered by checking the solvability of equation (204) at $i = j = 1$.
- If i is even and j is odd, i.e. $i = 2t$ and $j = 2r + 1$ for all positive integers $t, r \ge 1$, then equation (205) leads to

$$E_3 = 2(4^t - 1)^2 + 3(2 \cdot 4^r + 1)^2 + (4^{t+r+1} - 1)^2 \\ - 2(4^t - 1) \cdot (2 \cdot 4^r + 1) \cdot (4^{t+r+1} - 1) = 0.$$

  Again, we get a contradiction since $E_3 \pmod 8 \equiv 4$ for all $t, r \ge 1$. Here, we excluded all the even values of $i \ge 2$ and odd values of $j \ge 3$, and it remains to check whether equation (204) has solutions or not only at $j = 1, i \ge 2$. Similarly, this can be covered by studying the solutions of equation (204) only at $i = j = 1$.
- Finally, if i is odd and j is even, i.e. $i = 2t + 1$ and $j = 2r$ for all positive integers $t, r \ge 1$, then similarly we have

$$E_4 = 2(2 \cdot 4^t + 1)^2 + 3(4^r - 1)^2 + (4^{t+r+1} - 1)^2 \\ - 2(2 \cdot 4^t + 1) \cdot (4^r - 1) \cdot (4^{t+r+1} - 1) = 0,$$

and $E_4 \pmod 8 \equiv 4$ for all $t, r \geq 1$, which gives a contradiction. It is clear that all the odd values of $i \geq 3$ and even values of $j \geq 2$ are excluded, and we need to check whether equation (204) has solutions or not only at $i = 1, j \geq 2$.

From these cases, we conclude that it only remains to study the solutions of equation (204) at $i = 1$ and all the integers of j with $j \geq 1$. This can be done by direct substitution and using argument (III) as follows. It is clear that we have $k = i + j + 1 = j + 2$ and

$$2 + 3J_j^2 + J_{j+2}^2 - 6J_jJ_{j+2} = 0 \quad \text{for} \quad j \geq 1. \tag{206}$$

- If $j = 1$, then we have that $-4 = 2+3J_1^2+J_3^2-6J_1J_3 = 0$, which is impossible.
- If $j = 2$, then we get the solution $(i,j,k) = (1,2,4)$. Hence, equation (204) has the solution $(J_i, J_j, J_{i+j+1}) = (J_i, J_j, J_k) = (J_1, J_2, J_4) = (1,1,5)$.
- If $j \geq 3$, we can show that equation (204) has no more solutions by showing that
$$2 + 3J_j^2 + J_{j+2}^2 - 6J_jJ_{j+2} < 0 \quad \text{for} \quad j \geq 3.$$
Indeed, after substituting the Jacobsthal numbers formula $J_j = J_{j-1} + 2J_{j-2}$ in the left hand side of equation (206) a few times we get that
$$2 + 3J_j^2 + J_{j+2}^2 - 6J_jJ_{j+2} = 2 - 2J_{j-1}^2 - 24J_{j-1}J_{j-2} - 24J_{j-2}^2 < 0,$$
for $j \geq 3$, and this contradicts equation (206).

Therefore, by permuting the components of the solution $(1,1,5)$ to be a solution of equation (202) at the tuple $(1,2,3,6)$ we get the solution $(5,1,1)$. $\square$

Summary

This dissertation deals with some types of Diophantine equations related to linear recurrence sequences. In fact, it consists of three chapters. In Section 1.1, we introduce a historical survey of Diophantine equations (particularly, Fermat's equation). In Section 1.2, we recall some types of Diophantine equations with some related results, that appear briefly or in details with our main results in the other chapters. In Section 1.3, we recall some important notations, definitions and properties related to the subject of linear recurrence sequences, that will be used throughout the dissertation. Then we recite some recent results related to the solutions of some Diophantine equations involving linear recurrence sequences. In the final section of Chapter 1, we give an outline for our main results and the plan of the dissertation within the other chapters.

The new results are mainly described in Chapter 2 and Chapter 3 in which each has two main sections. Indeed, each section begins with a preface in which we recite some relevant related results of the Diophantine problem(s), that we investigate its (their) solutions in connection with terms of some linear recurrence sequences. These results have been published in the papers [**104, 105, 106, 107**] and accepted for publication in the papers [**108**, Mathematica Bohemica journal] and [**110**, Periodica Mathematica Hungarica journal].

For the sake of simplicity in presenting these results, we start by recalling some standard notations, definitions and properties concerning linear recurrence sequences (see e.g. Subsection 1.3.1 of Chapter 1). A sequence $\{G_n\}$ is called a linear recurrence relation of order k if the recurrence

$$G_{n+k} = a_1 G_{n+k-1} + a_2 G_{n+k-2} + \ldots + a_k G_n + f(n)$$

holds for all $n \geq 0$ with the coefficients $a_1, a_2, \ldots, (a_k \neq 0) \in \mathbb{C}$ and $f(n)$ a function depending on n only. If $f(n) = 0$ such a recurrence relation is called homogeneous, otherwise it is called nonhomogeneous.

For the homogeneous recurrence relation, the polynomial

$$F(X) = X^k - a_1 X^{k-1} - \ldots - a_k = \prod_{i=1}^{s} (X - \alpha_i)^{r_i} \in \mathbb{C}[X],$$

where $\alpha_1, \alpha_2, \ldots, \alpha_s$ and $r_1, r_2, ..., r_s$ are respectively the distinct roots of $F(X)$ and their corresponding multiplicities, is called the characteristic polynomial of $\{G_n\}$. Thus, if $F(X) \in \mathbb{Z}[X]$ has k distinct roots, then there exist constants $c_1, c_2, \ldots, c_k \in$

$\mathbb{Q}(\alpha_1, \alpha_2, \ldots, \alpha_k)$ such that

$$G_n = \sum_{i=1}^{k} c_i \alpha_i^n$$

holds for all the nonnegative values of n. If $k = 3$, then the sequence is called a ternary linear recurrence sequence. Most of the well known ternary linear recurrence sequences are the Tribonacci sequence and Berstel's sequence, which are defined by

$$\begin{aligned} &T_0 = T_1 = 0, T_2 = 1, \quad T_{n+3} = T_{n+2} + T_{n+1} + T_n && \text{for} \quad n \geq 0, \\ &B_0 = B_1 = 0, B_2 = 1, \quad B_{n+3} = 2B_{n+2} - 4B_{n+1} + 4B_n && \text{for} \quad n \geq 0, \end{aligned}$$

respectively. On the other hand, if $k = 2$, then $\{G_n\}$ represents a binary recurrence sequence. In the following we recall some types of binary linear recurrence sequences with their properties. Let P and Q be nonzero relatively prime integers and $U_n = U_n(P, Q)$ and $V_n = V_n(P, Q)$ be defined by the following recurrence relations with their initials:

$$\begin{aligned} &U_0 = 0, U_1 = 1, \quad U_n = PU_{n-1} - QU_{n-2} && \text{for} \quad n \geq 2, \\ &V_0 = 2, V_1 = P, \quad V_n = PV_{n-1} - QV_{n-2} && \text{for} \quad n \geq 2. \end{aligned}$$

The characteristic polynomial of the recurrences is given by

$$X^2 - PX + Q,$$

which has the roots

$$\alpha = \frac{P + \sqrt{D}}{2} \quad \text{and} \quad \beta = \frac{P - \sqrt{D}}{2},$$

with $\alpha \neq \beta, \alpha + \beta = P, \alpha \cdot \beta = Q$ and $(\alpha - \beta)^2 = D$, where D is called the discriminant such that $D = P^2 - 4Q$. The sequences $\{U_n\}$ and $\{V_n\}$ are called the (first and second kind) Lucas sequences with the parameters (P, Q), respectively, and the terms of these sequences are the generalized Lucas numbers. The terms of Lucas sequences of the first and second kind satisfy the identity

$$V_n^2 = DU_n^2 + 4Q^n. \tag{1}$$

Moreover, the Lucas sequences of the first and second kind can be respectively written by the following formulas, that are known as Binet's formulas.

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad V_n = \alpha^n + \beta^n \quad \text{for} \quad n \geq 0.$$

If the ratio $\zeta = \frac{\alpha}{\beta}$ is a root of unity, then the sequences $\{U_n\}$ and $\{V_n\}$ are said to be degenerate, and non-degenerate otherwise. Indeed, we mainly deal with non-degenerate linear recurrence sequences. Furthermore, the Lucas sequences for some values of P and Q have specific names such as the sequences of Fibonacci numbers, Pell numbers, Lucas numbers, Jacobsthal numbers and balancing numbers, which are given by

$$\begin{aligned}
F_0 = 0, F_1 = 1, \quad & F_n = F_{n-1} + F_{n-2} & \text{for} \quad n \geq 2,\\
P_0 = 0, P_1 = 1, \quad & P_n = 2P_{n-1} + P_{n-2} & \text{for} \quad n \geq 2,\\
L_0 = 2, L_1 = 1, \quad & L_n = L_{n-1} + L_{n-2} & \text{for} \quad n \geq 2,\\
J_0 = 0, J_1 = 1, \quad & J_n = J_{n-1} + 2J_{n-2} & \text{for} \quad n \geq 2,\\
B_0 = 0, B_1 = 1, \quad & B_n = 6B_{n-1} - B_{n-2} & \text{for} \quad n \geq 2,
\end{aligned}$$

respectively.

The goal of Section 2.1 is to extend the result of Tengely [**243**] in which he determined all the integer solutions (n, x) with $x \geq 2$ of the equation

$$\frac{1}{U_n(P,Q)} = \sum_{k=1}^{\infty} \frac{U_{k-1}(P,Q)}{x^k}.$$

We first determine the integral solutions (n, x) of the equation

$$\frac{1}{U_n(P_2,Q_2)} = \sum_{k=1}^{\infty} \frac{U_{k-1}(P_1,Q_1)}{x^k}, \tag{2}$$

for certain given pairs $(P_1, Q_1) \neq (P_2, Q_2)$. Here, we consider sequences with $1 \leq P \leq 3$ and $Q = \pm 1$. We also obtain the integral solutions (x, y) of the equation

$$\sum_{k=1}^{\infty} \frac{U_{k-1}(P,Q)}{x^k} = \sum_{k=1}^{\infty} \frac{R_{k-1}}{y^k}, \tag{3}$$

where the parameters of the Lucas sequence of the first kind represented by $1 \leq P \leq 3$ and $Q = \pm 1$, and the sequence $\{R_n\}$ is a ternary linear recurrence sequence represented by the Tribonacci sequence $\{T_n\}$ or Berstel's sequence $\{B_n\}$. Furthermore, we provide general results related to the integral solutions (x, y) of the equations

$$\sum_{k=1}^{\infty} \frac{U_{k-1}(P_1,Q_1)}{x^k} = \sum_{k=1}^{\infty} \frac{U_{k-1}(P_2,Q_2)}{y^k}, \tag{4}$$

with arbitrary pairs $(P_1, Q_1) \neq (P_2, Q_2)$, and

$$\sum_{k=1}^{\infty} \frac{T_{k-1}(a_2,a_1,a_0)}{x^k} = \sum_{k=1}^{\infty} \frac{T_{k-1}(b_2,b_1,b_0)}{y^k}, \tag{5}$$

where the triples $(a_2, a_1, a_0) \neq (b_2, b_1, b_0)$ and T_n denotes the general term of the generalized Tribonacci sequence that is given by

$$T_0(p,q,r) = T_1(p,q,r) = 0, T_2(p,q,r) = 1 \quad \text{and}$$

$$T_n(p,q,r) = pT_{n-1}(p,q,r) + qT_{n-2}(p,q,r) + rT_{n-3}(p,q,r),$$

for $n \geq 3$. Then we apply these results to completely resolve some concrete equations. Here, our main results also extend many former results obtained by e.g. Stancliff [**230**], Winans [**259**], Hudson and Winans [**120**], Long [**149**] and De Weger [**70**]. In order to introduce our new results of this section, we first recall the following. Let the set S be defined as follows

$$S = \{u_1(n) = U_n(1,-1), u_2(n) = U_n(2,-1), u_3(n) = U_n(3,-1),$$
$$u_4(n) = U_n(3,1)\}.$$

Moreover, in general we assume that the positive integers x, y in the investigated equations (2)–(5) satisfy the conditions of the following lemmas due to the results of Köhler [**135**]:

LEMMA. *Let A, B, a_0, a_1 be arbitrary complex numbers. Define the sequence $\{a_n\}$ by the recursion $a_{n+1} = Aa_n + Ba_{n-1}$. Then the formula*

$$\sum_{k=1}^{\infty} \frac{a_{k-1}}{x^k} = \frac{a_0 x - Aa_0 + a_1}{x^2 - Ax - B}$$

holds for all complex x such that $|x|$ is larger than the absolute values of the zeros of $x^2 - Ax - B$.

LEMMA. *Let arbitrary complex numbers $A_0, A_1, \ldots, A_m, a_0, a_1, \ldots, a_m$ be given. Define the sequence $\{a_n\}$ by the recursion*

$$a_{n+1} = A_0 a_n + A_1 a_{n-1} + \cdots + A_m a_{n-m}.$$

Then for all complex z such that $|z|$ is larger than the absolute values of all zeros of $q(z) = z^{m+1} - A_0 z^m - A_1 z^{m-1} - \cdots - A_m$, the formula

$$\sum_{k=1}^{\infty} \frac{a_{k-1}}{z^k} = \frac{p(z)}{q(z)}$$

holds with $p(z) = a_0 z^m + b_1 z^{m-1} + \cdots + b_m$, where $b_k = a_k - \sum_{i=0}^{k-1} A_i a_{k-1-i}$ for $1 \le k \le m$.

Then we prove the following results, that appear in the papers [**105, 106**].

THEOREM. *The equation*

$$\frac{1}{u_j(n)} = \sum_{k=1}^{\infty} \frac{u_i(k-1)}{x^k},$$

has the following solutions with $1 \le i, j \le 4, i \ne j$

$$(i,j,n,x) \in \{(1,2,1,2), (1,2,3,3), (1,2,5,6), (1,3,1,2), (1,3,5,11),$$
$$(1,3,7,35), (1,4,1,2), (1,4,5,8), (2,1,3,3), (2,1,9,7), (3,1,4,4), (3,$$
$$1,14,21), (3,4,2,4), (3,4,7,21), (4,1,\{1,2\},3), (4,1,5,4), (4,1,10,9),$$
$$(4,1,11,11), (4,2,1,3), (4,2,3,4), (4,2,5,7), (4,3,1,3), (4,3,5,12),$$
$$(4,3,7,36)\}.$$

THEOREM. *The complete list of solutions of the equation*

$$\sum_{k=1}^{\infty} \frac{u_j(k-1)}{x^k} = \sum_{k=1}^{\infty} \frac{R_{k-1}}{y^k},$$

with $u_n \in S, R_n \in \{B_n, T_n\}$ *and positive integers* x, y *is as follows*

u_n	R_n	(x,y)	u_n	R_n	(x,y)
u_1	B_n	$\{(25,9)\}$	u_1	T_n	$\{(2,2)\}$
u_2	B_n	$\{\}$	u_2	T_n	$\{(t(t^2-2)+1, t^2-1) : t \geq 2, t \in \mathbb{N}\}$
u_3	B_n	$\{(6,3),(18,7)\}$	u_3	T_n	$\{\}$
u_4	B_n	$\{(26,9)\}$	u_4	T_n	$\{(3,2)\}$

THEOREM. *Let* P_1, Q_1, P_2, Q_2 *be non-zero integers such that* $(P_1, Q_1) \neq (P_2, Q_2)$. *If* $(P_2^2 - P_1^2) + 4(Q_1 - Q_2) = d_1 d_2 \neq 0$ *and* $d_1 - d_2 \equiv -2P_1 \pmod 4$, $d_1 + d_2 \equiv -2P_2 \pmod 4$, *then the positive integral solutions* x, y *of*

$$\sum_{k=1}^{\infty} \frac{U_{k-1}(P_1,Q_1)}{x^k} = \sum_{k=1}^{\infty} \frac{U_{k-1}(P_2,Q_2)}{y^k}$$

satisfy

$$x = \frac{d_1 - d_2 + 2P_1}{4} > m(x^2 - P_1 x + Q_1), \quad y = \frac{d_1 + d_2 + 2P_2}{4} > m(x^2 - P_2 x + Q_2).$$

If $(P_2^2 - P_1^2) + 4(Q_1 - Q_2) = 0$ *and* $P_1 \equiv P_2 \pmod 2$, *then the positive integral solutions* x, y *of*

$$\sum_{k=1}^{\infty} \frac{U_{k-1}(P_1,Q_1)}{x^k} = \sum_{k=1}^{\infty} \frac{U_{k-1}(P_2,Q_2)}{y^k}$$

satisfy

$$x > m(x^2 - P_1 x + Q_1), \quad y = \pm x + \frac{P_2 \mp P_1}{2} > m(x^2 - P_2 x + Q_2),$$

where $Q_2 = Q_1 + \frac{P_2^2 - P_1^2}{4}$, *and* $m(f) = \max\{|x| : f(x) = 0$, *where* $f(x)$ *is a given polynomial over integers*$\}$.

THEOREM. *If* (x, y) *is an integral solution of the equation*

$$\sum_{k=1}^{\infty} \frac{T_{k-1}(a_2,a_1,a_0)}{x^k} = \sum_{k=1}^{\infty} \frac{T_{k-1}(b_2,b_1,b_0)}{y^k},$$

for given $(a_2, a_1, a_0) \neq (b_2, b_1, b_0)$, *then either*

$$9\left(a_2^2 - b_2^2 + 3\,a_1 - 3\,b_1\right)y + 2\,a_2^3 - 3\,a_2^2 b_2 + b_2^3 + 9\,a_1 a_2 - 9\,a_1 b_2 + 27\,a_0 - 27\,b_0 = 0$$

or in case of $|y| > B$ *we have*

$$|3x - 3y - a_2 + b_2| < C,$$

where B, C *are constants depending only on* $a_i, b_i, i = 0, 1, 2$.

As applications to the latter two theorems, we provide the following examples, that are described in [**106**].

EXAMPLE. *Let* $(P_1, Q_1) = (1, -1)$ *and* $(P_2, Q_2) = (18, 1)$, *then the solutions are as follows*

$$\begin{aligned}
\sum_{k=1}^{\infty} \frac{U_{k-1}(1,-1)}{2^k} = \sum_{k=1}^{\infty} \frac{U_{k-1}(18,1)}{18^k} &= 1,\\
\sum_{k=1}^{\infty} \frac{U_{k-1}(1,-1)}{7^k} = \sum_{k=1}^{\infty} \frac{U_{k-1}(18,1)}{20^k} &= \frac{1}{41},\\
\sum_{k=1}^{\infty} \frac{U_{k-1}(1,-1)}{10^k} = \sum_{k=1}^{\infty} \frac{U_{k-1}(18,1)}{22^k} &= \frac{1}{89},\\
\sum_{k=1}^{\infty} \frac{U_{k-1}(1,-1)}{15^k} = \sum_{k=1}^{\infty} \frac{U_{k-1}(18,1)}{26^k} &= \frac{1}{209},\\
\sum_{k=1}^{\infty} \frac{U_{k-1}(1,-1)}{26^k} = \sum_{k=1}^{\infty} \frac{U_{k-1}(18,1)}{36^k} &= \frac{1}{649},\\
\sum_{k=1}^{\infty} \frac{U_{k-1}(1,-1)}{79^k} = \sum_{k=1}^{\infty} \frac{U_{k-1}(18,1)}{88^k} &= \frac{1}{6161}.
\end{aligned}$$

EXAMPLE. *In case of* $(P_1, Q_1) = (1, -1)$ *and* $(P_2, Q_2) = (2t+1, t^2+t-1)$ *for some* $t \in \mathbb{Z}$, *we get that*

$$\sum_{k=1}^{\infty} \frac{U_{k-1}(1,-1)}{x^k} = \sum_{k=1}^{\infty} \frac{U_{k-1}(2t+1, t^2+t-1)}{(x+t)^k} = \frac{1}{x^2-x-1}$$

for $x \geq 2$.

EXAMPLE. *Consider the positive integral solutions* x, y *of the equation*

$$\sum_{k=1}^{\infty} \frac{T_{k-1}(-1,7,3)}{x^k} = \sum_{k=1}^{\infty} \frac{T_{k-1}(5,-5,-3)}{y^k}.$$

We obtain that the only integral solutions are given by

$$(x, y) \in \{(-1,1), (-3,3), (-2,4)\}.$$

Thus, we do not get positive integral solutions.

EXAMPLE. *Let us consider the equation*

$$\sum_{k=1}^{\infty} \frac{T_{k-1}(-4,-5,-6)}{x^k} = \sum_{k=1}^{\infty} \frac{T_{k-1}(1,8,18)}{y^k}.$$

Here, we get that the only positive solution is given by $(x, y) = (9, 11)$, *that is we have*

$$\sum_{k=1}^{\infty} \frac{T_{k-1}(-4,-5,-6)}{9^k} = \sum_{k=1}^{\infty} \frac{T_{k-1}(1,8,18)}{11^k} = \frac{1}{1104}.$$

EXAMPLE. *Finally, we provide an example in which we obtain infinitely many solutions. Let* $(a_2, a_1, a_0) = (1, 6, 5)$ *and* $(b_2, b_1, b_0) = (4, 1, 1)$. *Indeed, the integral solutions are given by* $(x, y) = (x, x+1)$ *for all* $x \geq 4$, *that is we have*

$$\sum_{k=1}^{\infty} \frac{T_{k-1}(1,6,5)}{x^k} = \sum_{k=1}^{\infty} \frac{T_{k-1}(4,1,1)}{(x+1)^k} = \frac{1}{x^3 - x^2 - 6x - 5}, \quad x \geq 4.$$

In Section 2.2, we firstly use a direct approach to obtain a general finiteness result for the Diophantine equation

$$G_n = B \cdot \Big(\frac{g^{lm} - 1}{g^l - 1}\Big), \tag{6}$$

where n, m, g, l and B are positive integers such that $m > 1, g > 1$, l is even, $1 \leq B \leq g^l - 1$, and G_n denotes the general term of an integer linear recurrence sequence represented by $U_n(P, Q)$ and $V_n(P, Q)$, with $Q \in \{-1, 1\}$. Indeed, the first finiteness result for equation (6), in case of $(G_n)_{n\geq1}$ is an integer linear recurrence sequence and l is a positive integer, was given by Marques and Togbé [**163**] in which they used heavy computations followed by a result due to Matveev [**166**] on the lower bound on linear forms of logarithms of algebraic numbers to obtain bounds for n and m. As these bounds could be very high, they used a result due to Dujella and Pethő [**76**] on the Baker-Davenport reduction to reduce these bounds. Then they applied this result to determine all the solutions of the Diophantine equations

$$F_n = B \cdot \Big(\frac{10^{lm} - 1}{10^l - 1}\Big) \quad \text{and} \quad L_n = B \cdot \Big(\frac{10^{lm} - 1}{10^l - 1}\Big) \tag{7}$$

in positive integers m, n and l, with $m > 1, 1 \leq l \leq 10$ and $1 \leq B \leq 10^l - 1$, which are $(m, n, l) = (2, 10, 1)$ and $(m, n, l) = (2, 5, 1)$ in the Fibonacci and Lucas cases, respectively. It is clear that these equations have solutions only with $l = 1$. Here, one may ask the following natural questions:

- Is there another approach that is easier to apply to such concrete equations?
- Do the equations in (7) have solutions in any base g other than 10, say $g \geq 2$, in the case of $l = 1$?

In fact, here we answer the above questions positively. More precisely, our approach of obtaining a general finiteness result for equation (6) is mainly based on producing biquadratic elliptic curves of the following form (from combining equation (6) with identity (1)),

$$y^2 = ax^4 + bx^2 + c,$$

with integer coefficients a, b, c and discriminant $\Delta = 16ac(b^2 - 4ac)^2 \neq 0$. First of all, the finiteness of the number of the integral points on the latter curve is guaranteed by Baker's result [**11**] presented by the following theorem and its best improvement concerning the solutions of elliptic equations over $\mathbb{Q}$, that is due to Hajdu and Herendi [**103**].

THEOREM. *If the polynomial on the right of the Diophantine equation*

$$y^2 = a_0x^n + a_1x^{n-1} + \ldots + a_n,$$

where $n \geq 3$ *and* $a_0 \neq 0, a_1, \ldots, a_n \in \mathbb{Z}$*, possesses at least three simple zeros, then all of its solutions in integers* x, y *satisfy*

$$\max(|x|, |y|) < \exp\exp\exp\{(n^{10n}H)^{n^2}\},$$

where $H = \max_{0\leq i\leq n} |a_i|$.

Also, the integral points of such curves can be determined using an algorithm implemented in Magma [**33**] as `SIntegralLjunggrenPoints()` (based on results obtained by Tzanakis [**249**]) or an algorithm described by Alekseyev and Tengely [**4**] in which they gave an algorithmic reduction of the search for integral points on such a curve by solving a finite number of Thue equations. As applications of our result, we apply our method on the sequences of Fibonacci numbers and Pell numbers that satisfy equation (6). Furthermore, with the first application we also generalize the result of Marques and Togbé in [**163**] in the case of Fibonacci numbers by determining all the solutions (n, m, g, B, l) of the equation

$$F_n = B \cdot \left(\frac{g^{lm} - 1}{g^l - 1}\right)$$

in case of $2 \leq g \leq 9$ and $l = 1$. Note that the case of Lucas numbers can be generalized similarly, therefore we omit the details of this case. More precisely, we use our approach in case where we have l is even, otherwise we follow the technique of Marques and Togbé in [**163**] of using the result of Matveev on linear forms in three logarithms and the result of Dujella and Pethő on the method of Baker-Davenport reduction. In fact, our main results here also extend other related results obtained by e.g. Luca [**151**] and Faye and Luca [**82**]. Before presenting our new results, it is important to mention the following remark:

REMARK. *Since a finiteness result for equation* (6) *in case of* $G_n = U_n$ *or* $G_n = V_n$ *can be obtained in a similar way, we only present and prove this result in detail in the case of* $G_n = U_n$ *and omit the proof of the remaining case.*

Here, we prove the following theorems, that are obtained in [**108**].

THEOREM. *Let* P *and* Q *be nonzero relatively prime integers with* $Q \in \{-1, 1\}$ *and* t *be a positive integer. If* $G_n = U_n(P, Q)$ *is non-degenerate and* $l = 2t$*, then the Diophantine equation* (6) *has finitely many solutions of the form* (n, m, g, B, l)*, which can be effectively determined.*

THEOREM. *If* $G_n = F_n$*, then the Diophantine equation* (6) *has the following solutions with* $2 \leq g \leq 9, l \in \{1, 2, 4\}$ *and* $1 \leq B \leq \min\{10, g^l - 1\}$.

$$(n, m, g, B, l) \in \{(4,2,2,1,1), (5,2,4,1,1), (6,2,3,2,1), (6,2,7,1,1),$$
$$(7,3,3,1,1), (8,2,6,3,1), (8,3,4,1,1), (5,2,2,1,2), (8,3,2,1,2),$$
$$(9,2,4,2,2), (9,2,2,2,4)\}.$$

Furthermore, suppose that $2 \leq g \leq 9, l = 2, 1 \leq B \leq \min\{5, g^l - 1\}$ *and* $G_n = P_n$*, then equation* (6) *has no more solutions other than* $(n, m, g, B, l) = (3, 2, 2, 1, 2)$.

In Section 3.1, we present a technique with which we can investigate the nontrivial integer solutions (X, Y, Z) of any equation (that has infinitely many integer solutions according to Mordell [**176**, page 111]) of the form

$$AX^2 + Y^r = C'Z^2,$$

for certain nonzero integers A, C' and r with $r > 1$ being odd and $(X, Y) = (L_n, F_n)$(or $(X, Y) = (F_n, L_n)$), where F_n and L_n denote the general terms of the sequences of Fibonacci numbers and Lucas numbers, respectively. We also remark that this technique can be applied on such equations for which they satisfy some conditions derived from a result due to Kedlaya [**128**] on solving constrained Pell equations. More precisely, we present the use of this technique for determining the solutions (X, Y, Z) of the Diophantine equation

$$7X^2 + Y^7 = Z^2, \tag{8}$$

where $(X, Y) = (L_n, F_n)$(or $(X, Y) = (F_n, L_n)$) and Z is a nonzero integer. From identity (1) (in case of $U_n(1, -1) = F_n$ and $V_n(1, -1) = L_n$), this technique shows that the solutions of equation (8) are equivalent to the solutions of the systems

$$x^2 - 5y^2 = \pm 4, \qquad 7x^2 + y^7 = z^2,$$
$$x^2 - 5y^2 = \pm 4, \qquad x^7 + 7y^2 = z^2,$$

where $x = L_n$, $y = F_n$ and $z = Z$ is a nonzero integer. More generally, a few techniques for investigating the integer solutions of certain systems of Diophantine equations of the form

$$x^2 - ay^2 = b, \qquad P(x, y) = z^2, \tag{9}$$

where a is a positive integer that is not a perfect square, b is a nonzero integer and $P(x, y)$ is a polynomial with integer coefficients, have been used by several authors such as Cohn [**57**] who considered the case where P is a linear polynomial. Cohn's method uses congruence arguments to eliminate some cases and a clever invocation of quadratic reciprocity to handle the remaining cases. The congruence arguments are very sufficient if there exists no solution in such a system, however they fail in the presence of a solution. This method was adapted by Mohanty and Ramasamy [**171**], Muriefah and Al Rashed [**1**], Peker and Cenberci [**182**] to study the solutions of particular systems. On the other hand, Kedlaya [**128**] gave a general procedure, based on the methods of Cohn and the theory of Pell equations, that solves many systems of the form (9). In fact, he applied this approach on several examples in which P is univariate with degree at most two. Moreover, in some cases this procedure fails to solve a system completely. Therefore, our technique mainly uses Kedlaya's procedure and similar techniques adapted by the methods of Mohanty and Ramasamy, Muriefah and Rashed, and Peker and Cenberci to prove the following theorems, that appear in [**104**].

THEOREM. *Suppose that* $X = L_n$ *and* $Y = F_n$*, then the Diophantine equation* (8) *has no more solutions other than* $(X, Y, Z) = (3, 1, \pm 8)$.

THEOREM. *The Diophantine equation* (8) *has no solutions in integers* X, Y *and* Z *if* $X = F_n$ *and* $Y = L_n$.

Finally, in Section 3.2 we present a technique for studying the solutions of some generalizations of Markoff equation in the numbers of certain binary linear recurrence sequences. In fact, Markoff equation is the Diophantine equation

$$x^2 + y^2 + z^2 = 3xyz$$

in positive integers $x \leq y \leq z$, which was deeply studied by Markoff [**160, 161**] in which he obtained many interesting results related to the solutions of this equation such as he showed that this equation has infinitely many integer solutions. The idea of investigating the solutions of the Markoff equation from some binary linear recurrence sequences was initiated by Luca and Srinivasan [**155**] in which they proved that the only solution of Markoff equation with $x \leq y \leq z$ such that $(x, y, z) = (F_i, F_j, F_k)$ is given by the well-known identity related to the Fibonacci numbers

$$1 + F_{2n-1}^2 + F_{2n+1}^2 = 3F_{2n-1}F_{2n+1}.$$

Therefore, in this section we present our new results in two subsections in which we extend the result of Luca and Srinivasan by simplifying their strategy with having upper bounds for the minimum of the indices to provide a direct approach for investigating such special solutions of the Jin-Schmidt equation and the Markoff-Rosenberger equation, that are respectively defined by

$$AX^2 + BY^2 + CZ^2 = DXYZ + 1, \tag{10}$$

where $(A, B, C, D) \in S$, with

$$S = \{(2,2,3,6), (2,1,2,2), (7,2,14,14), (3,1,6,6), (6,10,15,30), (5,1,5,5)\},$$

and

$$ax^2 + by^2 + cz^2 = dxyz, \tag{11}$$

where $(a, b, c, d) \in T$ such that

$$T = \{(1,1,1,1), (1,1,1,3), (1,1,2,2), (1,1,2,4), (1,1,5,5), (1,2,3,6)\}.$$

Equations (10) and (11) are clearly generalizations of the Markoff equation, that are due to the results of Jin and Schmidt in [**123**] and Rosenberger in [**198**], respectively. More precisely, in Subsection 3.2.1 we determine the solutions $(X, Y, Z) = (F_I, F_J, F_K)$ in positive integers of the Jin-Schmidt equation (10), where F_I denotes the I^{th} Fibonacci number. In other words, we study the solutions of the following Diophantine equations in the sequence of Fibonacci numbers:

$$2X^2 + 2Y^2 + 3Z^2 = 6XYZ + 1, \tag{12}$$

$$2X^2 + Y^2 + 2Z^2 = 2XYZ + 1, \tag{13}$$

$$7X^2 + 2Y^2 + 14Z^2 = 14XYZ + 1, \tag{14}$$

$$3X^2 + Y^2 + 6Z^2 = 6XYZ + 1, \tag{15}$$

$$6X^2 + 10Y^2 + 15Z^2 = 30XYZ + 1, \tag{16}$$

$$5X^2 + Y^2 + 5Z^2 = 5XYZ + 1, \tag{17}$$

where $X = F_I, Y = F_J$ and $Z = F_K$. Here, we prove the following theorem, that is obtained in [**107**].

THEOREM. *Let m be a positive integer greater than 1. If $(X,Y,Z) = (F_I, F_J, F_K)$ is a solution of equation (10) with $(A,B,C,D) \in S$, then the complete list of solutions is given by*

Eq.	*(A,B,C,D)*	$\{(X,Y,Z)\}$
(12)	$(2,2,3,6)$	$\{(1,1,1),(1,2,1),(1,2,3),(2,1,1),(2,1,3),$ $(F_{2m-1},F_{2m+1},1),(F_{2m+1},F_{2m-1},1)\}$
(13)	$(2,1,2,2)$	$\{(2,3,2),(2,5,2),(2,5,8),(8,5,2)\}$
(14)	$(7,2,14,14)$	$\{(1,2,1),(1,5,1),(3,2,1),(3,2,5)\}$
(15)	$(3,1,6,6)$	$\{(1,2,1),(3,2,1),(3,2,5)\}$
(16)	$(6,10,15,30)$	$\{(1,1,1),(1,2,1),(1,2,3)\}$
(17)	$(5,1,5,5)$	$\{\}$

In Subsection 3.2.2, we provide general results for the solutions $(x,y,z) = (R_i, R_j, R_k)$ of the Markoff-Rosenberger equation (11), where R_i denotes the i^{th} generalized Lucas number of first/second kind, i.e. $R_i = U_i$ or V_i. Then we apply the strategy of achieving these results to completely resolve concrete equations, e.g. we determine solutions containing only balancing numbers B_n and Jacobsthal numbers J_n, respectively. In other words, if $\mathfrak{T}$ is the set of all distinct tuples (a,b,c,d) derived from permuting the first three components of elements in T, then we prove the following results, that will appear in [**110**].

THEOREM. *Let $(a,b,c,d) \in \mathfrak{T}$, $P \geq 2, -P-1 \leq Q \leq P-1$ such that $Q \neq 0$, $D > 0$ and*

$$B_0 = \min_{I\in\mathbb{Z}} \left|\alpha^I - \frac{d}{c\sqrt{D}}\right|, \quad B_1 = \min_{I\in\mathbb{Z}} \left|\alpha^I - \frac{d}{c}\right|.$$

If $B_0 \neq 0$, then $B_0 \geq \alpha^{-4}$ and if $B_1 \neq 0$, then $B_1 \geq 0.17$. Furthermore, if $x = U_i, y = U_j$ and $z = U_k$ with $1 \leq i \leq j \leq k$ is a solution of (11) and $B_0 \neq 0$, then $i \leq 12$. If $x = V_i, y = V_j$ and $z = V_k$ with $1 \leq i \leq j \leq k$ is a solution of (11) and $B_1 \neq 0$, then $i \leq 7$.

Note that the cases where we have $B_1 = 0$ were completely studied in the proof of the above theorem. Thus, it remains to classify the cases satisfying $B_0 = 0$, the result is as follows.

PROPOSITION. *If $P \geq 2$, $-P-1 \leq Q \leq P-1$, $Q \neq 0$ and $D > 0$, then $B_0 \neq 0$ fulfills unless*

- $e = 1,\ P = 3, Q = 2,\ \alpha = 2,\ \sqrt{D} = 1,\ I = 0,$
- $e = 2,\ P = 3, Q = 2,\ \alpha = 2,\ \sqrt{D} = 1,\ I = 1,$
- $e = 2,\ P = 4, Q = 3,\ \alpha = 3,\ \sqrt{D} = 2,\ I = 0,$
- $e = 3,\ P = 5, Q = 4,\ \alpha = 4,\ \sqrt{D} = 3,\ I = 0,$
- $e = 4,\ P = 3, Q = 2,\ \alpha = 2,\ \sqrt{D} = 1,\ I = 2,$
- $e = 4,\ P = 6, Q = 5,\ \alpha = 5,\ \sqrt{D} = 4,\ I = 0,$
- $e = 4,\ P = 2, Q = -3,\ \alpha = 3,\ \sqrt{D} = 4,\ I = 0,$
- $e = 5,\ P = 7, Q = 6,\ \alpha = 6,\ \sqrt{D} = 5,\ I = 0,$
- $e = 5,\ P = 3, Q = -4,\ \alpha = 4,\ \sqrt{D} = 5,\ I = 0,$
- $e = 6,\ P = 4, Q = 3,\ \alpha = 3,\ \sqrt{D} = 2,\ I = 1,$
- $e = 6,\ P = 8, Q = 7,\ \alpha = 7,\ \sqrt{D} = 6,\ I = 0,$
- $e = 6,\ P = 4, Q = -5,\ \alpha = 5,\ \sqrt{D} = 6,\ I = 0,$

where $e = d/c$ *such that* $(a, b, c, d) \in \mathfrak{T}$.

Indeed, our results here also extend other related results obtained by e.g. Kafle, Srinivasan and Togbé [**125**] and Altassan and Luca [**5**]. As applications to the latter theorem, we prove the following results, that will also appear in [**110**].

THEOREM. *If* $(x, y, z) = (B_i, B_j, B_k)$ *is a solution of the equation*

$$ax^2 + by^2 + cz^2 = dxyz$$

and $(a, b, c, d) \in \{(1, 1, 1, 1), (1, 1, 1, 3), (1, 1, 2, 2), (1, 1, 2, 4), (1, 1, 5, 5), (1, 2, 3, 6)\}$, *then there is at most one solution given by* $x = y = z = B_1 = 1$.

THEOREM. *If* $(x, y, z) = (J_i, J_j, J_k)$ *is a solution of the equation*

$$ax^2 + by^2 + cz^2 = dxyz$$

and $(a, b, c, d) \in \{(1, 1, 1, 1), (1, 1, 1, 3), (1, 1, 2, 2), (1, 1, 2, 4), (1, 1, 5, 5), (1, 2, 3, 6)\}$, *then the complete list of solutions is given by*

(a, b, c, d)	*solutions*
$(1, 1, 1, 1)$	$\{(3, 3, 3)\}$
$(1, 1, 1, 3)$	$\{(1, 1, 1)\}$
$(1, 1, 2, 2)$	$\{\}$
$(1, 1, 2, 4)$	$\{(1, 1, 1), (1, 3, 1), (1, 3, 5), (3, 1, 1), (3, 1, 5), (3, 11, 1), (11, 3, 1)\}$
$(1, 1, 5, 5)$	$\{(1, 3, 1), (3, 1, 1)\}$
$(1, 2, 3, 6)$	$\{(1, 1, 1), (5, 1, 1)\}$

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List of talks

(1) Solutions of generalizations of Markoff equation from linear recurrences, 64^{th} Annual Online Meeting of the Australian Mathematical Society, University of New England, Australia, December 8–11, 2020.
(2) Solutions of a generalized Markoff equation in Fibonacci numbers, 9^{th} Interdisciplinary Doctoral Conference, Doctoral Student Association of the University of Pécs, Hungary, November 27–28, 2020.
(3) Cryptanalysis of ITRU, 1^{st} Conference on Information Technology and Data Science, Faculty of Informatics, University of Debrecen, Hungary, November 6–8, 2020.
(4) Cryptanalysis of ITRU, 20^{th} Central European Conference on Cryptology, Department of Mathematics, Faculty of Science, University of Zagreb, Croatia, June 24–26, 2020.
(5) Diophantine equations related to reciprocals of linear recurrence sequences, 24^{th} Central European Number Theory Conference, J. Selye University, Komárno, Slovakia, September 1–6, 2019.
(6) Solutions of the Diophantine equation $7X^2 + Y^7 = Z^2$ from recurrence sequences. Institute of Mathematics, University of Debrecen, Hungary, April 12, 2019.
(7) Representations of reciprocals of Lucas sequences, CSM-The 5^{th} Conference of PhD Students in Mathematics, Bolyai Institute, University of Szeged, Hungary, June 25–27, 2018.

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Registry number: DEENK/59/2021.PL
Subject: PhD Publication List

Candidate: Hayder Raheem Hashim

Doctoral School: Doctoral School of Mathematical and Computational Sciences

List of publications related to the dissertation

Foreign language scientific articles in Hungarian journals (1)

1. **Hashim, H. R.**, Tengely, S.: Representations of reciprocals of Lucas sequences.
   Miskolc Math. Notes. 19 (2), 865-872, 2018. ISSN: 1787-2405.
   DOI: http://dx.doi.org/10.18514/MMN.2018.2520
   IF: 0.468

Foreign language scientific articles in international journals (5)

2. **Hashim, H. R.**, Tengely, S.: Lucas sequences and repdigits.
   Math. Bohem. "Accepted by Publisher", 1-18, 2021. ISSN: 0862-7959.
3. **Hashim, H. R.**, Szalay, L., Tengely, S.: Markoff-Rosenberger triples and generalized Lucas sequences.
   Period. Math. Hung. "Accepted by Publisher", 1-16, 2020. ISSN: 0031-5303.
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   Notes Numb. Theor. Discret. Math. 25 (2), 49-56, 2019. ISSN: 1310-5132.
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List of other publications

Foreign language scientific articles in international journals (7)

7. **Hashim, H. R.**, Molnár, A., Tengely, S.: Cryptanalysis of ITRU.
Rad HAZU Mat. Znan. [Epub], 1-13, 2021. ISSN: 1845-4100.

8. Alkufi, M. A. H. J., **Hashim, H. R.**, Hussein, A. M., Mohammed, H. R.: An algorithm based on GSVD for image encryption.
Math. Comput. Appl. 22 (2), 1-8, 2017. ISSN: 1300-686X.
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Int. J. Comput. Trends Technol. 50 (2), 94-100, 2017. ISSN: 2349-0829.
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Total IF of journals (all publications): 1,815
Total IF of journals (publications related to the dissertation): 1,815

The Candidate's publication data submitted to the iDEa Tudóstér have been validated by DEENK on the basis of the Journal Citation Report (Impact Factor) database.

19 February, 2021

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Nyilvántartási szám: DEENK/59/2021.PL
Tárgy: PhD Publikációs Lista

Jelölt: Hashim, Hayder Raheem
Doktori Iskola: Matematika- és Számítástudományok Doktori Iskola

A PhD értekezés alapjául szolgáló közlemények

Idegen nyelvű tudományos közlemények hazai folyóiratban (1)

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További közlemények

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A közlő folyóiratok összesített impakt faktora: 1,815
A közlő folyóiratok összesített impakt faktora (az értekezés alapjául szolgáló közleményekre): 1,815

A DEENK a Jelölt által az iDEa Tudóstérbe feltöltött adatok bibliográfiai és tudománymetriai ellenőrzését a tudományos adatbázisok és a Journal Citation Reports Impact Factor lista alapján elvégezte.

Debrecen, 2021.02.19.