Integral points here and there

Number Theory Seminar

26-10-2018

Additive Erdős-Graham problem

M. Ulas and Sz. Tengely: Power values of sums of certain products of consecutive integers and related results, J. of Number Theory, in press.

Let n be a non-negative integer and put

$$p_n(x) = \prod_{i=0}^{n}(x+i).$$

Consider the Diophantine equation

$$y^m = p_n(x) + \sum_{i=1}^{k} p_{a_i}(x),$$

where $m \in \mathbb{N}_{\geq 2}$ and $a_1 < a_2 < \ldots < a_k < n$. This equation can be considered as a generalization of the Erdős-Selfridge Diophantine equation $y^m = p_n(x)$.

Additive Erdős-Graham problem

We define the set

$$A_n = \{(a_1, \ldots, a_k) \in \mathbb{N}_0^k : \; a_i < a_{i+1} \text{ for } i = 1, 2, \ldots k-1, \; a_k < n \text{ and } k \in \{1, \ldots, n-1\}\}.$$

For given $m \in \mathbb{N}_{\geq 2}$ and $T = (a_1, \ldots, a_k) \in A_n$ we consider the Diophantine equation

$$y^m = g_T(x), \quad \text{where} \quad g_T(x) := p_n(x) + \sum_{i=1}^{k} p_{a_i}(x).$$

The cardinality of A_n is $2^n - 1$, hence for a given m we deal with $2^n - 1$ Diophantine equations.

Additive Erdős-Graham problem

The literature of this type of Diophantine equations is very extensive.

- Erdős and Rigge independently proved that a product of two or more consecutive integers is never a perfect square.
- Erdős and Selfridge proved that the above equation has no solutions in integers (x, y, m, n) satisfying the conditions $n \geq 1, m \geq 2$ and $y \neq 0$.
- Euler proved that a product of four terms in arithmetic progression is never a square.
- Obláth obtained a similar statement in case of five terms.
- Bennett, Bruin, Győry and Hajdu extended this result to the case of arithmetic progressions having at most 11 terms.
- Győry, Hajdu and Pintér extended these results for at most 34 terms.
- Hirata-Kohno, Laishram, Shorey and Tijdeman completely solved the Diophantine equations related to the cases of arithmetic progressions of length $3 \leq k < 110$.

Additive Erdős-Graham problem

- Bennett and Siksek in a recent paper shoved that if k is large enough, the equation in question has only finitely many solutions.
- Note that the additive equation, in the case $T = (0, 1, \ldots, n-1) \in A_n$, was studied in a recent paper of Hajdu, Laishram and Tengely. They proved that for $n \geq 1$ and $m \geq 2$ (with $n \neq 2$ in case of $m = 2$) the equation has only finitely many integer solutions. Moreover, they were also able to solve the equation explicitly for $n \leq 10$.

Additive Erdős-Graham problem

Based on general results by Tijdeman and Brindza we give finiteness statements.

Theorem
Let $n \in \mathbb{N}_{\geq 2}$, $T = (a_1, \ldots, a_k) \in A_n$. If $a_1 \geq 2$ or $a_1 = 1, a_2 = 3, a_3 \geq 5$ then for the integer solutions of the Diophantine equation $y^m = g_T(x)$ we have:

- if $y \neq 0$, then $m < c_1(n)$,
- if $m \geq 3$, then $\max\{m, |x|, |y|\} < c_2(n)$,
- if $m = 2$, then $\max\{|x|, |y|\} < c_3(n)$.

Conjecture
Let $n \in \mathbb{N}$ and $T \in A_n$ be given. The polynomial $g_T(x)$ has multiple roots if and only if:

1. $T = (n-4)$ for $n \geq 4$ with $(x^2 + (2n-3)x + n^2 - 3n + 1)^2 | g_T(x)$,
2. $T = (n-3, n-2)$ for $n \geq 3$ with $(x + n - 1)^3 | g_T(x)$,
3. $T = (n-2, n-1)$ for $n \geq 2$ with $(x + n)^2 | g_T(x)$.

In each of the above cases the corresponding co-factors have no multiple roots.

Additive Erdős-Graham problem

We were able to determine the set of rational points on the curves $y^2 = g_T(x)$ with $T \in A_n, n \leq 5$, except in the following cases:

T	$r(T)$	$C_T(\mathbb{Q})$
(2)	2	$\{(-2,0), (-1,0), (0,0), (2/3, \pm 460/3)\}$
(3)	2	$\{(-9, \pm 252), (-3,0), (-2,0), (-1,0), (0,0), (-1,0), (0,0), (-18/5, \pm 468/5)\}$
(0, 4)	4	$\{(0,0), (1, \pm 29), (9/4, \pm 5871/4)\}$
(1, 3)	2	$\{(-4, \pm 6), (-1,0), (0,0)\}$
(1, 4)	2	$\{(-1,0), (0,0)\}$
(0, 1, 2)	2	$\{(-2,0), (0,0), (1, \pm 27)\}$
(0, 2, 3)	2	$\{(0,0)\}$
(1, 2, 3)	2	$\{(-3,0), (-1,0), (0,0)\}$
(1, 3, 4)	2	$\{(-4, \pm 6), (-1,0), (0,0), (-25/9, \pm 620/9)\}$
(0, 1, 2, 3)	3	$\{(-4,0), (-2,0), (-1,0), (0,0), (-13/3, \pm 91/3), (-5/3, \pm 55/3)\}$
(0, 1, 3, 4)	2	$\{(-3,0), (-1,0), (0,0)\}$

The most difficult one seems to be the case with $T = (0,4)$. The genus 2 curve is given by $y^2 = x(x^5 + 16x^4 + 95x^3 + 260x^2 + 324x + 145)$. The rank of the Jacobian is 4 and one needs to work over a degree 5 number field.

Additive Erdős-Graham problem

We resolved the Diophantine equations $y^2 = g_T(x)$ for $T \in A_5, A_7, A_9, A_{11}$ and A_{13}. In all cases $g_T(x)$ is a monic polynomial, hence Runge's condition is satisfied. We note that in case of $T \in A_{13}$ there are $2^{13} - 1$ equations to be solved and the bounds obtained by Runge's method are of size 10^6. An improved reduction algorithm were used to make the computations feasible.

We also note that equations for which $\gcd(m, n+1) \geq 2$ can be solved using Runge's method. For example if $n = 14$ and $T = (10, 11, 12, 13)$, then we have

$$y^m = \left(x^3 + 39\,x^2 + 504\,x + 2157\right)(x + 12)p_{10}(x),$$

an equation that can be solved using Runge's method for $m = 3, 5$ and 15. We note that in all cases only the trivial solutions with $y = 0$ exist. In this way we were able to determine all solutions of equations with $(m, n) \in \{(5,3), (8,3), (11,3), (4,5), (9,5), (6,7)\}$.

Additive Erdős-Graham problem

Question
Let us consider the equation $y^m = g_T(x)$ in unbounded many variables $m \in \mathbb{N}_{\geq 2}, n \in \mathbb{N}_{\geq 2}$, $T \in A_n$, where m, n are chosen in such a way that the genus of the curve defined by our equation is positive. Is the set of positive integer solutions infinite?

x	$[m,\ n,\ T]$
1	$[2,4,(0)], [2,5,(0,4)], [2,5,(0,1,2)], [2,6,(0)], [2,6,(3,4)], [2,7,(0,3,4,5,6)],$ $[2,8,(0,3,7)], [2,8,(0,1,2,5)], [2,9,(0,1,2,5,6,7], [2,14,(0,1,2,6,\ldots,13)]$ $[3,5,(0,1,2)], [5,3,(1,2)], [7,4,(1,2)]$
2	$[2,3,(2)], [2,5,(2,3)], [7,3,(0,1)]$
4	$[2,6,(0,4)]$

$$\text{if } x=1: y^m = \sum_{i=1}^{n}(a_i+1)! \text{ in non-negative integers } a_1, a_2, \ldots \text{ and } y, m \in \mathbb{N}.$$

Additive Erdős-Graham problem

Theorem
The Diophantine equation $z^3 = p_2(x) + p_2(y)$ has infinitely many solutions (x, y, z) in polynomials with integer coefficients and satisfying $\deg_t x = \deg_t y$.

We have the factorization
$p_2(x) + p_2(y) = (x + y + 2)(x^2 - xy + y^2 + x + y)$. Write $z = 3t^2(x + y + 2)$, where t is a variable taking integer values,

$$U^2 - 3(108t^6 - 1)V^2 = 12(2916t^6 - 135t^6 + 1),$$

where

$$U = 3(108t^6 - 1)(x + 1), \quad V = (54t^6 + 1)x + 2(27t^6 - 1)y + 108t^6 - 1$$

or equivalently

$$x = \frac{U}{3(108t^6 - 1)} - 1, \quad y = \frac{3(108t^6 - 1)V - (54t^6 + 1)U}{6(27t^6 - 1)(108t^6 - 1)} - 1.$$

Additive Erdős-Graham problem

In the range $1 \leq x \leq y \leq 10^5$, equation $z^2 = p_2(x) + p_2(y)$ has 619 integer solutions. This relatively large number suggests the existence of a polynomial solution. We were tried quite hard to construct parametric solutions but we failed. This motivates us to formulate the following problem.

Question
Does the equation $z^2 = p_2(x) + p_2(y)$ has a solution in polynomials with integer coefficients?

Theorem
Let $i \in \{2, 3, 4\}$. The equation $z^2 = p_i(x) + p_i(y)$ has infinitely many solutions in positive integers.

Cyclic cubic fields

Shanks - "simplest cubic fields":

$$S_n = X^3 + (n+3)X^2 + nX - 1.$$

Family given by Lecacheux:

$$L_n = X^3 - (n^3 - 2n^2 + 3n - 3)X^2 - n^2X - 1.$$

Other family given by Kishi:

$$K_n = X^3 - n(n^2 + n + 3)(n^2 + 2)X^2 - (n^3 + 2n^2 + 3n + 3)X - 1.$$

Cyclic cubic fields

Steve Balady (PhD thesis, 2017) pointed out that these families are coming from solutions related to equation $X(3)$:

$$X(3): \quad x^3 + y^3 + z^3 = \lambda xyz.$$

Shanks - $[x : y : z; \lambda] = [0 : -1 : 1; n]$

Lecacheux - $[x : y : z; \lambda] = [-1 : -n : 1; -n^2]$

Kishi - $[x : y : z; \lambda] = [-n : -n^2 - n - 1 : 1; -n^3 - 2n^2 - 3n - 3]$

Key observation: $\lambda = \frac{f^3+g^3+1}{fg}$ is a polynomial in $\mathbb{Z}[X]$, where $f, g \in \mathbb{Z}[X]$ such that $\deg f \leq \deg g$.

Cyclic cubic fields

New family obtained by Balady with $(f, g) = (-n^2, n^3 - 1)$:

$$B_n = X^3 + (n^7 + 2n^6 + 3n^5 - n^4 - 3n^3 - 3n^2 + 3n + 3)X^2 + (-n^4 + 3n)X - 1.$$

Theorem (Balady)
If (f, g) provides a family, then $(g, \frac{g^3+1}{f})$ does too.

As an example he gives:

$$(f, g) = (-n^2, n^3 - 1), \quad (g, k_1) = (n^3 - 1, -n^7 + 3n^4 - 3n),$$

$$(k_1, k_2) = (-n^7 + 3n^4 - 3n, -n^{18} + \ldots).$$

Cyclic cubic fields

Ulas - Tengely: looking solutions of the following form

$$f(t) = \sum_{i=0}^{m-2} a_i t^i + a_m t^m, \quad g(t) = \sum_{i=0}^{n} b_i t^i.$$

Let

$$\bar{A} = (a_0, a_1, \ldots, a_{m-1}, a_m), \quad \bar{B} = (b_0, b_1, \ldots, b_{n-1}, b_n)$$

be the vectors of variables. We define the $F_i = F_i(\bar{A}, \bar{B}), i = 0, \ldots n-1$ and $G_i(\bar{A}, \bar{B}), i = 0, \ldots, m-1$ as the numerators of the coefficients in the remainders of divisibility of $f^3 + 1 \pmod{g}$ and $g^3 + 1 \pmod{f}$ respectively.

Cyclic cubic fields

We have

$$(f(t)^3+1) \pmod{g(t)} = \sum_{i=0}^{n-1} F_i(\bar{A},\bar{B})t^i, \quad g(t)^3+1 \pmod{f(t)} = \sum_{i=0}^{m-1} G_i(\bar{A},\bar{B})t^i.$$

The system of non-linear equations to consider:

$$S(m,n) : \begin{cases} F_i(\bar{A},\bar{B}) = 0, & i = 0,\dots,n-1, \\ G_j(\bar{A},\bar{B}) = 0, & j = 0,\dots,m-1. \end{cases}$$

Cyclic cubic fields

We have three non-trivial solutions in the case $m = 1$. These are given by

$$f(t) = t, \quad g(t) = -t - 1,$$
$$f(t) = t, \quad g(t) = -t^2 + t - 1,$$
$$f(t) = t, \quad g(t) = -t^3 - 1.$$

Cyclic cubic fields

If $\deg f = 2$, then one has to deal with the systems $S(2, n)$ for $n \in \{2, \ldots, 6\}$.

$$S(2,2): \begin{cases} F_0 = a_0^3 b_2^5 - 3a_0^2 a_2 b_0 b_2^4 + 3a_0 a_2^2 b_0^2 b_2^3 - a_2^3 b_0^3 b_2^2 - 3a_0 a_2^2 b_0 b_1^2 b_2^2 + 3a_2^3 b_0^2 b_1^2 b_2 - a_2^3 b_0 b_1^4 + b_2^5 = 0, \\ F_1 = a_2 b_1 (a_2^2 b_1^4 + 3a_0 a_2 b_2^2 b_1^2 - 4a_2^2 b_0 b_2 b_1^2 + 3a_0^2 b_2^4 - 6a_0 a_2 b_0 b_2^3 + 3a_2^2 b_0^2 b_2^2) = 0, \\ G_0 = a_2^3 b_0^3 - 3a_0 a_2^2 b_0 b_1^2 - 3a_0 a_2^2 b_0^2 b_2 + 3a_0^2 a_2 b_0 b_2^2 + 3a_0^2 a_2 b_1^2 b_2 - a_0^3 b_2^3 + a_2^3 = 0, \\ G_1 = b_1 (3a_2^2 b_0^2 - 6a_0 a_2 b_2 b_0 - a_0 a_2 b_1^2 + 3a_0^2 b_2^2) = 0. \end{cases}$$

Theorem
The only non-trivial solution of the system S(2,2) is given by

$$f(t) = \frac{1}{2}(t^2 - t + 1), \quad g(t) = \frac{1}{2}(t^2 + t + 1) = f(-t).$$

Cyclic cubic fields

Theorem
The only non-trivial solution of the system S(2,3) is given by

$$f(t) = -t^2, \quad g(t) = t^3 - 1.$$

Theorem
The only non-trivial solution of the system S(2,4) is given by

$$f(t) = \frac{1}{2}(t^2 - t + 1), \quad g(t) = \frac{1}{4}(t^2 + t + 1)(t^2 - t + 3).$$

Theorem
The only non-trivial solution of the system S(2,5) is given by

$$f(t) = -t^2 + t - 1, \quad g(t) = t(t^4 - 2t^3 + 4t^2 - 3t + 3).$$

Cyclic cubic fields

Hessian form of an elliptic curve:

$$H_d : \quad x^3 + y^3 + dxy + 1 = 0$$

Theorem
If $(x, y) \in \mathbb{Z}^2$ is a solution of equation H_d for some $|d| > 3$, then $|x| < |d| + 1$.

Runge's method can be applied. E.g. if $d = 3t + s$, then we get

$$\begin{aligned}(3x + 3y - 3t - s)(9x^2 - 9xy + 9y^2 + 3(3t + s)x + 3(3t + s)y + (3t + s)^2) = \\ = -(3t + s + 3)(9t^2 + 6ts + s^2 - 9t - 3s + 9).\end{aligned}$$

Cyclic cubic fields

10	$[(7,-4),(-4,7),(-1,0),(0,-1)]$
17	$[(-1,0),(0,-1),(-9,-7),(-7,-9)]$
22	$[(-1,0),(0,-1),(-9,-4),(-4,-9)]$
24	$[(7,-2),(-2,7),(-1,0),(0,-1)]$
57	$[(-1,0),(0,-1),(-3,-13),(-13,-3)]$
72	$[(-9,26),(26,-9),(-1,0),(0,-1)]$
90	$[(-1,0),(0,-1),(-9,-28),(-28,-9)]$
95	$[(36,-13),(-13,36),(-1,0),(0,-1)]$
111	$[(-1,0),(0,-1),(-21,-4),(-4,-21)]$
129	$[(-1,0),(0,-1),(-63,-37),(-37,-63)]$
140	$[(-1,0),(0,-1),(-18,-49),(-49,-18)]$
155	$[(-1,0),(0,-1),(-45,-76),(-76,-45)]$
159	$[(103,-56),(-56,103),(-1,0),(0,-1)]$
193	$[(545,-481),(-481,545),(-1,0),(0,-1),(-5,-31),(-31,-5)]$
205	$[(-7,38),(38,-7),(-1,0),(0,-1)]$
207	$[(-1,0),(0,-1),(-84,-37),(-37,-84)]$
244	$[(63,-16),(-16,63),(-1,0),(0,-1),(-81,-28),(-28,-81)]$

Cyclic quartic fields

Balady - Washington:

$$X_L: \quad (x+y)^4 - 4x^2y^2 + 4Lxy(x+y) + 4 = 0,$$

they have considered the special cases $L = \pm 2$:

$$\left(x^2 + 4\,xy + y^2 \mp 2\,x \mp 2\,y + 2\right)\left(x^2 + y^2 \pm 2\,x \pm 2\,y + 2\right) = 0.$$

Cyclic quartic family:

$$X^4 + (4n^3 - 4n^2 + 8n - 4)X^3 + (-6n^2 - 6)X^2 + 4X + 1.$$

Cyclic quartic fields

As in case of cubic fields we reduce the problem to a system of equations $R(m, n)$ having the coefficients of the polynomials as variables.
The case $R(1, 4)$. We can easily parametrize b_0, b_1, b_2, b_3 as follows

$$\begin{aligned} b_0 &= \frac{a_0^4 b_4 + 4b_4}{a_1^4}, \\ b_1 &= \frac{4a_0^3 b_4}{a_1^3}, \\ b_2 &= \frac{6a_0^2 b_4}{a_1^2}, \\ b_3 &= \frac{4a_0 b_4}{a_1}. \end{aligned}$$

Cyclic quartic fields

Hence we obtain the parametrization

$$\begin{aligned} b_0 &= \pm(1/4a_0^4+1),\\ b_1 &= \pm a_0^3 a_1,\\ b_2 &= \pm 3/2a_0^2a_1^2,\\ b_3 &= \pm a_0 a_1^3,\\ b_4 &= \pm\frac{a_1^4}{4}. \end{aligned}$$

As an example fix $a_0 = a_1 = 2$, we get the family of quartic polynomials

$$X^4 + 4(n+1)(4n^6+24n^5+54n^4+56n^3+29n^2+10n+1)X^3+ \\ +6(2n^4+8n^3+10n^2+4n+1)X^2+4(n+1)^3X+1.$$

Cyclic quartic fields

Theorem
If $L = 2n$, then let $D_1 = \{d : d|(4n^4 - 4)\}$. In this case we have that

$$2(x+y)^2 = d_1 + d_2 - 4n^2,$$

where $d_1, d_2 \in D_1$ and $d_1 d_2 = 4n^4 - 4$. If $L = 2n + 1$, then let $D_2 = \{d : d|(16n^4 + 32n^3 + 24n^2 + 8n - 15)\}$. We have that

$$(2x+2y)^2 = d_1 + d_2 - 2(2n+1)^2,$$

where $d_1, d_2 \in D_1$ and $d_1 d_2 = 16n^4 + 32n^3 + 24n^2 + 8n - 15$.

Cyclic quartic fields

Theorem
If $5 \leq L \leq 10^6$, then the only non-trivial solutions are

$$L = 10: \quad (x, y) \in \{(-5, -1), (-1, -5)\},$$
$$L = 19309: \quad (x, y) \in \{(-5, 629), (629, -5)\}.$$

The degree 4 polynomials are as follows

$$x^4 + 148x^3 + 102x^2 + 20x + 1,$$
$$x^4 + 7890798742x^3 - 37333446x^2 + 38618x + 1.$$

Thank you for your attention

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