Faculty of Science and Technology
Institute of Mathematics

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Masters of Science Thesis

Integral Graphs

by:

Mayala Lutome Patrick

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Supervisor: Dr. Szabolcs Tengely
Associate Professor

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Contents

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Sincerely,
Mayala Lutome Patrick
April 24, 2020

1. Introduction

Let G be a graph with the vertex set $\{v_1, \ldots, v_n\}$. The adjacency matrix of G is an $n \times n$ matrix $A(G)$ whose (i, j)-entry is 1 if v_i is adjacent to v_j and is 0, otherwise. The zeros of the characteristic polynomial of $A(G)$ are called the eigenvalues of G. Throughout this paper all graphs under consideration are simple (a finite undirected graph without loops or multiple edges).
The graph G is said to be *integral if all the eigenvalues of G are integers.* The notion of integral graphs was first introduced by Harary and Schwenk in 1974 [12]. There are many simple examples of integral graphs. For example in [19];

- Trees $(T_{(1;2)}, T_{(1;6)}, T_{(3,1)})$.
- The complete multipartite graph $(K_{m_1,m_2,\ldots,m_k})$.
- The path graph and in this case only P_2 is the integral path in the set of paths P_n with n vertices.
- The circuits C_3, C_4 and C_6 (the three circuits are the only integral circuits in the set of circuits C_n with n vertices).
- The stars $K_{1,n}$ with $n = p^2$ $(p = 1, 2, 3, \ldots)$.

Integral graph being a modern field in mathematics it has recently been discovered that integral graphs may be of interest for designing the network topology of perfect state transfer networks [9, 14].
The spectrum of G is the set of eigenvalues of A [7]. The spectrum of the adjacency matrix of a graph G is known as *adjacency spectrum of G* or simply, spectrum of G. Since the spectrum of a disconnected graph is the union of the spectra of its components, in any investigation of integral graphs it is sufficient to consider connected graphs only.
Generally the problem of characterizing integral graphs seems to be very difficult. Since there is no general characterization (besides the definition) of these graphs. The problem of finding (or characterizing) integral graphs has to be treated in some special interesting families of integral graphs for example; trees, regular graphs, complete multipartite graphs, complete split-like graphs, connected cubic integral Cayley graphs, integral regular graphs e.t.c [17].

However despite difficulties of its characterization, integral graphs is closely related to other graphs such as Laplacian integral and Gaussian digraphs.

Laplacian integral: Given a graph G on n vertices, its Laplacian matrix is the $n \times n$ matrix L given by $Lap(G) = D(G) - A(G)$; where A is the $(0,1)$ adjacency matrix, and $D(G)$ is the diagonal matrix of vertex degrees. Graphs with integral Laplacian eigenvalues are called *Laplacian integral.* Integral graphs and Laplacian integral graphs, have differences but in some cases are related.
For example: Let G be r-regular graph, then $Lap(G) + A(G) = rI$. So μ is an eigenvalue of $Lap(G)$ if and only if $r - \mu$ is an eigenvalue of $A(G)$. This means that a regular graph is Laplacian integral if and only if it is integral [10, 21].
Some graph operations, which can be applied to integral graphs, when applied to Laplacian case give rise to integral graphs.
For example: The complete product of graphs, being the complement of the disjoint union (direct sum) of their complements.

$$G1 \nabla G2 := \overline{(\overline{G1} \cup \overline{G2})}$$

Therefore the complete product of Laplacian graphs form an integral graph.

Gaussian digraphs: Consider digraphs, contrary to (non-oriented) graphs, whose spectra are real, the eigenvalues of digraphs are complex numbers.
Note that the adjacency matrix $A(G)$ of a digraph need no more be symmetric.
A complex number $\lambda = \alpha + i\beta$ is called a *Gaussian integer* if α and β are integers.
A digraph is called *Gaussian* if its spectrum consists only of *Gaussian* integers and if all of them are real integers, such a digraph will be called integral. As for integral digraphs, we note that there is an interesting example of two cospectral integral digraphs with four vertices, which are the smallest integral digraphs [6]. For any positive integer n we can find n cospectral strongly connected non-symmetric digraphs which are integral [6].

2. Basics of integral graphs

2.1 Construction of Integral graphs

Constructions and properties of integral graphs are among the major focus to many researchers, this is due to the fact that it is difficult to construct and to give general characterization of all integral graphs [19]. However, graph operations such as Cartesian product, Strong sum and Product on integral graphs can be used for constructing finitely many families of integral graphs [4].
Let G and H be two graphs with vertex sets $V(G)$ and $V(H)$. The above three operations define graphs having $V(G) \times V(H)$ as its vertex set [4].
If λ_i, $(i = 1, 2 \ldots, n)$ and $\mu_j, (j = 1, 2, \ldots, m)$ are the eigenvalues of G and H, respectively.

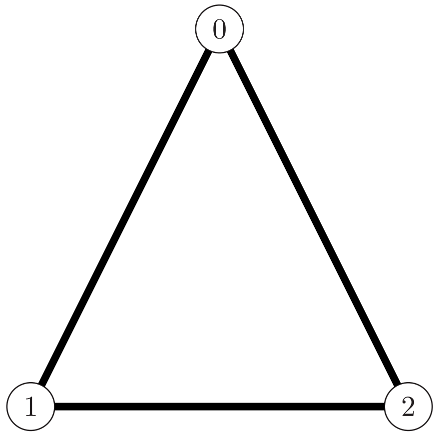
For example : Consider the following two integral graphs G and H.
Where; G = Path graph (P_2) and H = Complete graph (K_3).

Path graph (P_2).

$$G = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Has eigenvalues $\lambda_i = [1, -1]$.

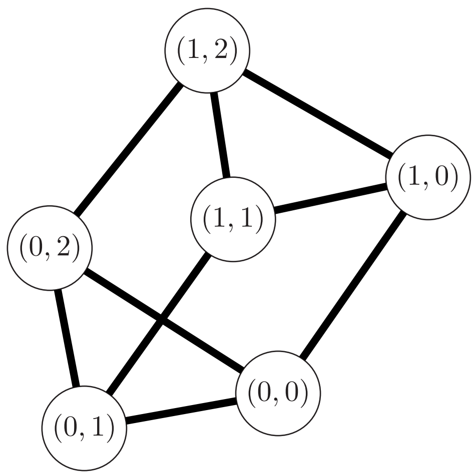
Complete graph (K_3).



$$H = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

Has eigenvalues $\mu_j = [2, -1, -1]$.

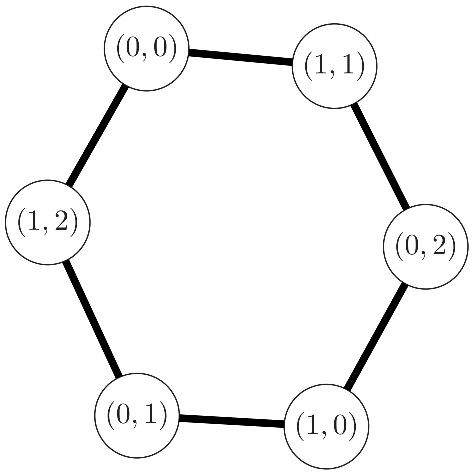
1. The *sum (* or *Cartesian product)* of G and H is given as $(G \times H)$.



$$M = \begin{pmatrix} 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \end{pmatrix}$$

Has eigenvalues $\lambda_i + \mu_j = [3, 1, 0, 0, -2, -2]$.

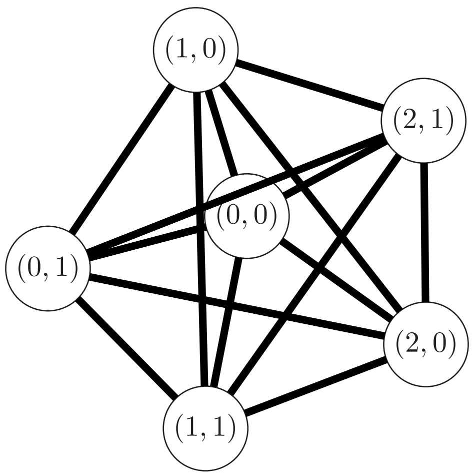
2. The *Product (*or *Conjunction) $G \times H$* of G and H.



$$M = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Has eigenvalues $\lambda_i \mu_j = [2, -2, 1, 1, -1, -1]$.

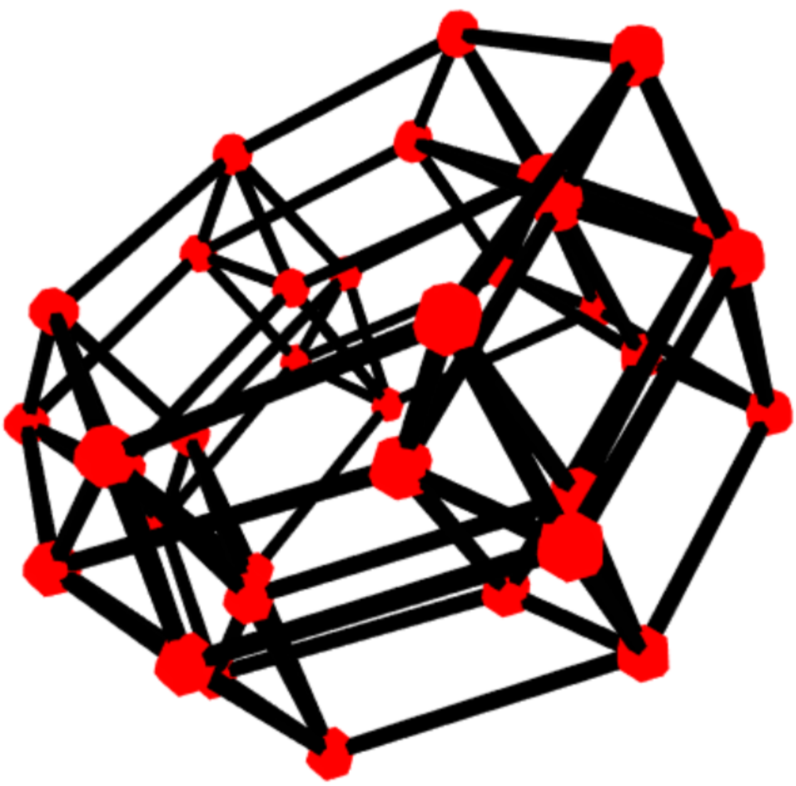
3. The *strong sum (*or *strong product) $G \oplus H$* of G and H.



$$M = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

Has eigenvalues $\lambda_i\mu_j + \lambda_i + \mu_j = [5, -1, -1, -1, -1, -1]$.

Interestingly, these graph products when the operations are applied to them they produces again integral graphs. Therefore with these operations we can produce finitely many integral graphs.
For example:
Let $M =$ The *sum* or *Cartesian product* of G and H is given as $(G \times H)$.
Let $N =$ The *Product* or *Conjunction* i.e $G \times H$ of G and H.
Then, the *sum (*or *Cartesian product)* of M and N is given as $(M \times N)$.



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\end{pmatrix}$$

Has eigenvalues: $\lambda_i \mu_j + \lambda_i + \mu_j = \{5,\ 3,\ 4,\ 4 - 2,\ -2,\ -4 - 4,\ 0,\ 0,\ 0,\ 0,\ -3,\ 3,\ -3,\ -3,\ 1,\ 1,\ 1,\ 1,\ 1,\ 2,\ 2,\ 2,\ 2,\ 2,\ 2,\ -1,\ -1,\ -1,\ -1,\ -1,\ -1,\ -1,\ -1,\ -1,\ -1\}$; $(\forall i = 1, 2, \ldots, n \quad and \quad \forall j = 1, 2, \ldots, m)$.

Thus, these three operations preserve the integrality. Other integrality preserving graph operations can be found in [12], such as the complementary graph (or the line graph) of an integral regular graph G.

2.2 Eigenvalues of graphs

Definition 2.2.1. An *eigenvalue* (λ) of a graph G is an eigenvalue of its adjacency matrix $A(G) \in R^{|V(G)|}$. That is $\lambda \in R$ for which there is an eigenvector $v \in R^{|V(G)|}$, $v \neq 0$ such that :

$$A(G)v = \lambda v.$$

The vector v is called an eigenvector corresponding to λ.

In most cases the eigenvalues of a graph are distinct, but when many eigenvalues coincide, then it appears that we are in a very special case.

- If all eigenvalues are the same, then we must have an *empty graph*.
- If we have only two eigenvalues, then essentially we must have a *complete graph*.

Properties of eigenvalues

(a) All eigenvectors corresponding to λ form a subspace V_λ; the dimension of V_λ is called the *multiplicity* of λ and the sum of all eigenvalues, including multiplicities, is the trace of the matrix:

$$\sum_{i=1}^{n} \lambda_i = Tr(A).$$

(b) The product of all eigenvalues, each taken with the same multiplicity as it occurs among the roots, is the determinant of the matrix:

$$\prod_{i=1}^{n} \lambda_i = det(A).$$

(c) The number of non-zero eigenvalues, including multiplicities, is the rank of A.

Theorem 2.2.1. *(Spectral Theorem).*

(i) The eigenvalues of a graph G are always real.

(ii) The adjacency matrix $A(G)$ is diagonalizable.

(iii) There is an orthonormal basis of eigenvectors.

Lemma 2.2.1. *The sum of all eigenvalues of a graph is always* 0.

Lemma 2.2.2. *The largest eigenvalue λ_1 of a graph G lies between the average and maximum degrees: $d_{avg} \leq \lambda_1 \leq d_{max}$. In particular, if G is k-regular, then $\lambda_1 = k$ [2].*

Proof. First we need to prove that $\lambda_1 \leq d_{max}$.
Let $x = (x_v), v \in V(G)$ be an eigenvector corresponding to λ_1.
Let x_u be the entry of x with maximum absolute value. Then we have (with $N(u) = v \in V(G) : uv \in E(G)$ the neighbourhood of u)

$$\lambda_1 x_u = \sum_{v \in N(u)} x_v,$$

so (with $du = |N(u)|$ the degree of u)

$$|\lambda_1| \leq \sum_{v \in N(u)} |x_v| \leq \sum_{v \in N(u)} |x_u| \leq d_u \cdot |x_u| \leq d_{max} \cdot |x_u|.$$

Since $x \neq 0$, so we have $|x_u| \neq 0, \lambda = |\lambda_1| \leq d_{max}$ (we have $\lambda_1 > 0$, since $0 = T_r(A) = \sum \lambda_i$).

To prove that $d_{avg} \leq \lambda_1$. Consider $j^T A j$.
On the one hand $j^T A j = \sum\limits_{v \in V(G)} d_v = 2|E(G)|$.
On the other hand, take an orthonormal basis $\{v_1, \ldots, v_n\}$ of eigenvectors of A, and let $j = \sum c_i v_i$. The representation of j in this basis. So we have $Av_i = \lambda_i v_i$, $j^T v_i = c_i$ and $\sum c_i^2 = \| j \|^2 = n$. Then;
$j^T A j = \sum c_i j^T (Av_i) = \sum c_i j^T (\lambda_1 v_i) = \sum \lambda_i c_i (j^T v_i) = \sum \lambda_i c_i^2 < \lambda_1 \sum c_i^2 = \lambda_1 n.$
So we get $\lambda_1 \geq 2|E(G)|/n = d_{avg}$. □

Lemma 2.2.3. *(Connectedness). If G is k-regular, then the multiplicity of the eigenvalue λ_1 equals the number of connected components of G.*

Lemma 2.2.4. *(Diameter). If G is connected, then the diameter of G is strictly less than its number of distinct eigenvalues.*

Lemma 2.2.5. *(Bipartiteness). A graph is bipartite if and only if its spectrum is symmetric (i.e if λ is an eigenvalue, then so is $-\lambda$, and with the same multiplicity) [2].*

Proof. First suppose G is bipartite, with parts S and T of sizes s and t. This means that for some $s \times t$ matrix B, we have;

$$A = \begin{pmatrix} 0_{ss} & B \\ B^T & 0_{tt} \end{pmatrix}.$$

If λ is an eigenvalue then;

$$\begin{bmatrix} \lambda v \\ \lambda w \end{bmatrix} = \lambda \begin{bmatrix} v \\ w \end{bmatrix} = A \begin{bmatrix} v \\ w \end{bmatrix} = \begin{bmatrix} Bw \\ B^T v \end{bmatrix}.$$

So $Bw = \lambda v$ and $B^T v = \lambda w$. Then $-\lambda$ is also an eigenvalue:

$$A \begin{bmatrix} v \\ -w \end{bmatrix} = \begin{bmatrix} -Bw \\ B^T v \end{bmatrix} = \begin{bmatrix} -\lambda v \\ \lambda w \end{bmatrix} = -\lambda \begin{bmatrix} v \\ w \end{bmatrix}$$

If λ has multiplicity m, then so does $-\lambda$, since the subspace spanned by the $[v\ w]^T$ will have the same dimension as that spanned by the corresponding $[v\ \ -w]^T$. This means that the spectrum is symmetric. □

2.3 Characterization of integral graphs

Integral graphs as other graphs can be characterized or classied into various forms. In this section we will discuss characterization of integral graphs when they are regular and complete multipartite graphs.

2.3.1 Regular graphs

Definition 2.3.1. Let $k \geq 2$ be an integer. We say that the graph G is $k-$regular if for every $v_i \in V : \sum_{v_j \in V} A_{ij} = k$ [7].

If G is simple (*has neither parallel edges nor loop*), this amounts to saying that each vertex has exactly k neighbours. Assume that G is a finite graph on n vertices. Then A is an $n \times n$ symmetric matrix; hence, it has n real eigenvalues, counting multiplicities, that we may list in decreasing order: $\lambda_0 \geq \lambda_1 \geq, \ldots, \geq \lambda_{n-1}$ and for every eigenvalue λ_i we have $|\lambda_i| \leq k$ [2, 7].

Regular graphs have square adjacency matrix . A matrix is said to be regular if all its row sums and column sums are equal. The common value of the row and column sum is called the regularity of the matrix.
For a graph G, let $L(G)$ denote the line graph of G, in which $V(L(G)) = E(G)$, and where two vertices are adjacent if and only if they are edges with common endpoint in G. If a regular graph G is integral, then its line graph are not only integral but also Laplacian integral [1].

Lemma 2.3.1. *If G is a regular graph of degree k, then its line graph $L(G)$ is regular of degree $2k-2$.*

Lemma 2.3.2. *If G is a regular graph of degree k with n vertices and $m = \frac{1}{2}nk$ edges, then;*

$$P(L(G), x) = (x+2)^{\frac{1}{2}n(k-2)} P(G, x+2-k).$$

Lemma 2.3.3. *If G is a connected $k-$regular graph on n vertices with four distinct eigenvalues, then;*

1. *G has four integral eigenvalues, or*
2. *G has two integral eigenvalues, and two eigenvalues of the form $\frac{1}{2}(a \pm \sqrt{b})$, with $a, b \in Z$, $b > 0$, with the same multiplicity, or*
3. *G has one integral eigenvalue, its degree k, and the other three have the same multiplicity $m = \frac{1}{3}(n-1)$, and $k = m$ or $k = 2m$.*

There are many integral graphs which are regular; *For example; Complete graphs.* A complete graph is a graph with n vertices and an edge between every

two vertices. In older literature, complete graphs are sometimes called universal graphs. The complete graph is $(n-1)$ regular [18]; as it was presented in previous section some operations on a regular graph provide some infinite families of integral graphs.
Example: Consider a complete graph with 10 vertices (K_{10}). This is 9-regular graph its adjacency matrix and spectrum are as follows:-

$$M = \begin{pmatrix} 0&1&1&1&1&1&1&1&1&1\\ 1&0&1&1&1&1&1&1&1&1\\ 1&1&0&1&1&1&1&1&1&1\\ 1&1&1&0&1&1&1&1&1&1\\ 1&1&1&1&0&1&1&1&1&1\\ 1&1&1&1&1&0&1&1&1&1\\ 1&1&1&1&1&1&0&1&1&1\\ 1&1&1&1&1&1&1&0&1&1\\ 1&1&1&1&1&1&1&1&0&1\\ 1&1&1&1&1&1&1&1&1&0 \end{pmatrix}.$$

Its spectrum is $[9,-1,-1,-1,-1,-1,-1,-1,-1,-1]$ which contains only integer eigenvalues.

2.3.2 Complete multipartite graphs

A multipartite graph G is called complete multipartite (or complete k-partite) if each vertex of V_i is adjacent to each vertex of V_j ($i \neq j = 1, 2, \ldots, k$). Thus in a complete multipartite graph any two vertices are adjacent if and only if they belong to two distinct subsets [5, 11]. A complete k-partite graph is generally denoted by $K_{m_1,m_2,\ldots,m_k}$ [11]. The total number of vertices and edges in a complete multipartite graph are given by $|V| = \sum m_i$ and $|E| = \sum m_i m_j$.

Let the number of distinct integers of $m_1, m_2, m_1, m_3, \ldots, m_k$ be r. Without loss of generality, assume that the first r ones are the distinct integers such that $m_1 < m_2 < m_3 < \ldots < m_r$. Suppose that b_i is the multiplicity of m_i for each $i = 1, 2, \ldots, r$.
The complete k-partite graph $K_{m_1,m_2,\ldots,m_k} = K_{m_1\ldots m_1,\ldots,m_r\ldots m_r}$ is also denoted by $K_{b_1m_1,b_2m_2,\ldots b_rm_r}$. Where $k = \sum_{i=1}^{r} b_i$ and $|V| = n = \sum_{i=1}^{r} b_i m_i$.
For example: The complete 2-partite graph $K_{m_1m_2}$ (i.e. $r = 2$ and $b_1 = b_2 = 1$) is integral if and only if $m_1 m_2$ is a perfect square [19].

Theorem 2.3.1. *The complete k-partite graph* $(K_{m_1,m_2,\ldots,m_k}) = K_{b_1\cdot m_1,b_2\cdot m_2,b_3\cdot m_3,\ldots b_r\cdot m_r}$, *is integral if and only if;*

$$\prod_{i=1}^{r}(x+m_i) - \sum_{j=1}^{r} b_j m_j \prod_{i=1,i\neq j}^{r}(x+m_j) = 0 \tag{2.3.1}$$

has only integral roots [19].

Considering the equation 2.3.1 to get more information. Firstly, we divide both sides of Equation by $\prod_{i=1}^{k}(x+m_i)$ we obtain;

$$F(x) := \sum_{i=1}^{r} \frac{b_i m_i}{x+m_i} - 1 = 0. \tag{2.3.2}$$

It is trivial that $-m_i$ is not a root of *Equation (2.3.1)*, for $1 \leq i \leq r$. Hence, the solutions of *Equation (2.3.1)* are the same as those of *Equation (2.3.2)*. Now we consider the roots of $F(x)$ over the set of real numbers. Note that $F(x)$ is discontinuous at each point $-m_i$. For $1 \leq i \leq r$, we have that $F(m_i - 0) = -\infty$,

$$F(m_i+0) = +\infty, F(-\infty) = F(+\infty) = -1,\ F'(x) = -\sum_{i=1}^{r} \frac{b_i m_i}{(x+m_i)^2}. \tag{2.3.3}$$

Clearly $F(x)$ is strictly monotone decreasing on each of the intervals, where $F(x)$ is continuous. Using the Weierstrass Intermediate Value Theorem of Analysis, we deduce that $F(x)$ has distinct real roots,

$$-\infty < u_r < u_{r-1} < m_r < u_r < -m_{r-1} < u_{r-1} < \ldots < -m_2 < u_2 < -m_1 < 0 < u_1 < +\infty. \tag{2.3.4}$$

On other hand *Equation (2.3.2)* can be written as;

$$\frac{m_1 b_1}{x+m_1} + \frac{m_2 b_2}{x+m_2} + \frac{m_3 b_3}{x+m_3} \ldots, + \frac{m_r b_r}{x+m_r} = 1 \ \ [19]. \tag{2.3.5}$$

For example: Consider a complete multipartite graph $K_{5,8,12}$.
It has a spectrum $[16,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ -6,\ -10]$ in which we see that all eigenvalues are integers.
In case of graphs in this type with $(a < b < c)$ the eigenvalues are the same as the following matrix:

$$M = \begin{pmatrix} 0 & b & c \\ a & 0 & c \\ a & b & 0 \end{pmatrix}.$$

In our example $a = 5, b = 8$ and $c = 12$. therefore its matrix is:-

$$M = \begin{pmatrix} 0 & 8 & 12 \\ 5 & 0 & 12 \\ 5 & 8 & 0 \end{pmatrix}.$$

It has eigenvalues $[16, -10, -6]$. That is the same as above, except the multiple eigenvalue 0.
By Cardano's formula we obtain that the matrix has integral eigenvalues if $(ab + ac + bc)^3 - 27(abc)^2$ is a square. If we look for special solutions satisfying some extra conditions, then we may find some families.

The identification of all integral graphs appears to be hopelessly involved. However, as with many other problems in graph theory, if we restrict our attention to trees, the prospects are much better as one .

Chapter 3 of this thesis will present integral trees as an example of integral graphs. In this chapter will describe a construction of *integral trees* starting with integral trees of diameter 3, generalization of integral trees with even diameter and finally generalization of integral tree with odd diameter. In the discussion some conditions on the parameters of integral trees will be developed in order to generate infinite families of integral trees.

3. Integral trees

3.1 Introduction

A *tree* is a connected undirected graph without cycles. An *integral tree* is a tree for which the eigenvalues of its adjacency matrix are all integers. Among the early results about integral trees is that by Watanabe [15] that says that an integral tree different from K_2 does not have a complete matching. Another early interesting result was *starlike* integral trees, that is all integral trees with at most one vertex of degree larger than 2, were given in [20].
In recent years many different classes of integral trees have been constructed and each class contain finitely many integral trees. The following are examples of integral trees: $T_{(1;2)}, T_{(1;6)}, T_{(3,1)}$ [12].

3.2 Integral Trees and Pell's equation

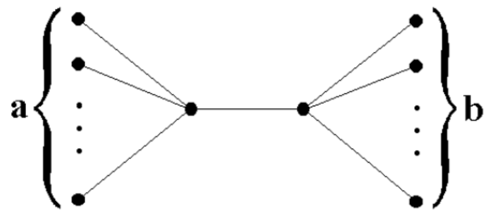
In this section, we revisit integral trees of diameter 3 discussed by Pokorny in[16]. The tree of diameter 3 is balanced, if all the vertices of the same distance from the center are of the same degree and is denoted by $T(1;n_1)$. It is well known that a balanced tree $T = T(1;n_1)$ is integral tree if and only if $n_1 = s_1(s_1+1)$ where $s_1 \in N$ [16]. The main concern of this section is to give a relationship between integral trees of diameter 3 and Pell's equations and construct integral trees of diameter 3 using them.

We will use the theory of divisors and codivisors, which is useful in the spectral graph theory. The most important property of divisor D of a graph G is that the characteristic polynomial $P(D;x)$ divides the characteristic polynomial of G , i.e. there exists a polynomial $P(C;x)$ such that:-

$$P(G;x) = P(D;x) \cdot P(C;x).$$

Generally a divisor D of a graph G is a directed graph with multiple edges and loops and its codivisor C is a directed graph whose arcs are valued by plus or minus one.

Every tree of diameter 3 can be characterized by two parameters $a, b \in N$ and denoted by $T(1; a|b)$.



Proposition 1. The characteristic polynomial of the corresponding divisor $D(1; a|b)$ of the tree $T(1; a|b)$ can be expressed by the formula;

$$P(D(1; a|b)) = x^4 - (a + b + 1)x^2 + ab. \tag{3.2.1}$$

Proposition 2. The characteristic polynomial of a corresponding codivisor $C(1; a|b)$ of the tree $T(1; a|b)$ can be expressed by the formula;

$$P(C(1; a|b)) = x^{a+b-2}. \tag{3.2.2}$$

Proposition 3. The characteristic polynomial of the tree $T(1; a|b)$ can be expressed by the formula;

$$P(D(1; a|b)) = (x^4 - (a + b + 1)x^2 + ab)x^{a+b-2}. \tag{3.2.3}$$

Hence the spectrum of codivisor $C(1; a|b)$ consists only of eigenvalues. Now, it is sufficient to find positive integers a, b so that the equation (3.2.1) has only integer eigenvalues.

If we want the tree $T(1; a|b)$ to be integral, then equation (3.2.1) has to have only integer eigenvalues. A spectrum of a tree is always symmetric, therefore $Sp(D(1; a|b)) = \{\pm r, \pm s\}$, where $r, s \in N$. We have to find $a, b \in N$ for which the equation (3.2.1) has integer eigenvalues.
Consider

$$P(D(1; a|b)) = x^4 - (r^2 + s^2)x^2 + r^2 \cdot s^2. \tag{3.2.4}$$

Comparing equation (3.2.1) and equation (3.2.4)

$$a \cdot b = r^2 \cdot s^2 \tag{3.2.5}$$

$$a + b + 1 = r^2 + s^2. \tag{3.2.6}$$

If we eliminate r using equation (3.2.5) and (3.2.6), then;

$$(a - s^2)(b - s^2) = s^2. \tag{3.2.7}$$

Let $a - s^2 = s_1$ and $b - s^2 = s_2$, then using equations (3.2.6) and (3.2.7) it is clear that:-

$$s_1 \cdot s_2 = s^2 \tag{3.2.8}$$

$$s_1 + s_2 + 1 = r^2 - s^2 \tag{3.2.9}$$

$$(1 + s_1)(1 + s_2) = r^2. \tag{3.2.10}$$

We need to find solutions $s_1, s_2 \in N$ of equations (3.2.8) and (3.2.10).
It is easy to show that if $s_1 = s_2$, then $a = b = s_1(s_1 + 1)$, where $s_1 \in N$ and $T(1; a|b)$ is balanced tree of diameter 3 [13].

Let s_1 be arbitrary positive integer, then from equation (3.2.8), we have;

$$s_2 = \frac{s^2}{s_1}.$$

Denote by h the greatest common divisor of s and s_1. Let $s = h \cdot s'$. Then $s^2 = h^2 \cdot (s')^2$. Now let $s_1 = h \cdot s_1'$. Since $h = (s, s_1)$ we have $(s', s_1') = 1$.
Since

$$s_2 = \frac{h^2 s'^2}{h s_1{}'} = \frac{h s'^2}{s_1{}'}$$

and $(s', s_1{}') = 1$, then we have $s_1{}'|h$ and $h = s' \cdot s_1{}''$.

Using equation (3.2.9) it is easy to show that;

$$s_1 + s_2 + 1 = r^2 - s^2 = r^2 - h^2 s'^2$$

$$s_1 + 1 = r^2 - \frac{s_1{}' h^2 + h}{s_1{}'} s'^2$$

and

$$s_1 + 1 = r^2 - (h^2 + s_1{}'') s'^2. \tag{3.2.11}$$

Since s_1 is the positive integer, $s_1{}'$ and h are divisors of s_1 and $s_1 = h \cdot s_1{}'$ then,

$$h = s_1{}' \cdot s_1{}'' \tag{3.2.12}$$

hence

$$s_1 = ({s_1}')^2 \cdot {s_1}''. \tag{3.2.13}$$

That is why (3.2.11) is the Pell's equation with form $r^2 - D \cdot {s'}^2 = L$,
where $D = h^2 + {s_1}''$ and $L = {s_1}' + 1$.
In order to find the solutions of equations (3.2.8) and (3.2.10) we have to solve the the Pell's equation (3.2.11).

Theorem 3.2.1. *The tree $T(1; a|b)$ is integral one and its spectrum is* $\left\{ \pm r, \pm s, \underbrace{0, \ldots, 0}_{a+b-2} \right\}$ *if and only if the following formulas hold:-*

$$a = s_1 + s^2 \qquad\qquad b = s_2 + s^2$$

$$s_2 = \frac{s^2}{s_1} \qquad\qquad s = h \cdot s'.$$

(r, s_1') is the solution of the equation (3.2.11) where s_1 is arbitrary positive integer and both equations (3.2.12) and (3.2.13) holds for h, s_1''.
The proof of the necessary condition is above Theorem 3.2.1. Conversely, using formulas from the Theorem (3.2.1) in equation (3.2.1) it is easy to show by calculus that the tree $T(1; a|b)$ is integral.

Corollary 1. *Let $s_1 = 1$. Using (3.2.12) and (3.2.13) it can be verified that the equation (3.2.11) has the form;*

$$2 = r^2 - 2{s'}^2. \tag{3.2.14}$$

Then;

$$2 = r^2 - 2{s'}^2 \implies r^2 = 2 + 2{s'}^2.$$

Let $s' = 1$ then $(r, s') =$ $(2, 1)$ hence; $2 + \sqrt{2}$ is the solution.
For

$$1 = r^2 - 2{s'}^2 \implies r^2 = 1 + 2{s'}^2$$

Let $s' = 2$ then $(r, s') =$ $(3, 2)$ hence; $3 + 2\sqrt{2}$ is the fundamental solution.
Therefore all its solutions can be expressed by the formula;

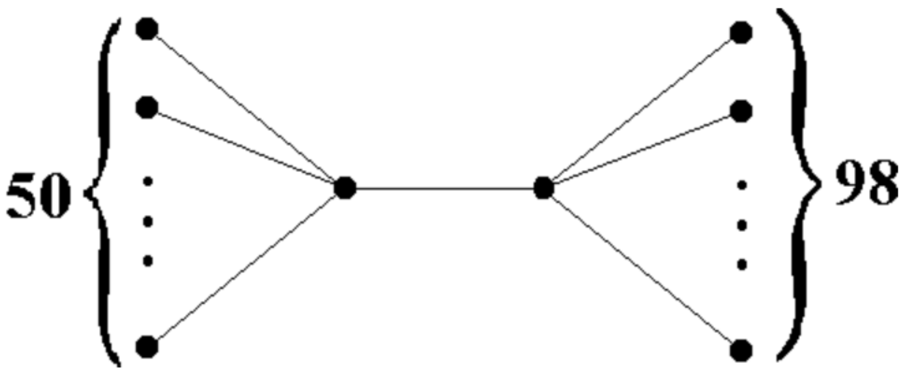
$$r_n + s_n'\sqrt{2} \quad = \quad (2 + \sqrt{2})(3 + 2\sqrt{2})^n$$

where; $n = 0, 1, 2, 3, 4, \ldots$.

From Theorem 3.2.1 above: The values of a and b are given as follows; $a = {s'}^2 + 1$ and $b = 2{s'}^2$.

For example:
If $n = 0$ then $r_0 + s'_0\sqrt{2} \quad = \quad (2+\sqrt{2})(3+2\sqrt{2})^0$, hence; $(r_0, s'_0) = \ (2,1)$ and $a = b = 2$.
We have a balanced integral tree $T(1; 2|2)$.

If $n = 1$ then $r_1 + s'_1\sqrt{2} \quad = \quad (2+\sqrt{2})(3+2\sqrt{2})$, hence; $(r_1, s'_1) = \ (10,7)$ and $a = 50, \ \ b = 98$.



We obtain a non balanced tree $T(1; 50|98)$, which has the smallest number of vertices from all trees of this class. The characteristics polynomial of its divisor is;

$$P_D = x^4 - (149)x^2 + 4900 \quad = (x^2 - 100)(x^2 - 49)$$

and its spectrum is $S_D = \{\pm 10, \pm 7\}$.

If $n = 2$ then $r_2 + s'_2\sqrt{2} \quad = \quad (2+\sqrt{2})(3+2\sqrt{2})^2$, hence $(r_1, s'_1) = \ (58,41)$ is the solution of its ordered pair and $a = 1682, \ \ b = 3362$.
We obtain a non balanced tree $T(1; 1682|3362)$. The characteristics polynomial of its divisor is

$$P_D = x^4 - (5045)x^2 + 5654884 \quad = (x^2 - 3364)(x^2 - 1681) = (x^2 - 58^2)(x^2 - 41^2)$$

and its spectrum is $S_D = \{\pm 58, \pm 41\}$.

Corollary 2. *Let* $s_1 = 2$. *Using the same method as in Corollary 1.*

$$3 = r^2 - 6s'^2.$$

Then;

$$3 = r^2 - 6s'^2 \implies r^2 = 3 + 6s'^2.$$

Let $s' = 1$ then; $(r, s') =\ (3, 1)$ hence; $3 + \sqrt{6}$ is the solution.
For

$$1 = r^2 - 6s'^2 \implies r^2 = 1 + 6s'^2.$$

Let $s' = 2$ then $(r, s') =\ (5, 2)$ hence; $5 + 2\sqrt{6}$ is the fundamental solution.
Therefore all its solutions can be expressed by the formula;

$$r_n + s'_n\sqrt{6} \quad = \quad (3 + \sqrt{6})(5 + 2\sqrt{6})^n$$

where n = 0, 1, 2, 3, 4 , ...

From Theorem 3.2.1 above: The values of a and b are given as follows;

$$a = (2s'_n)^2 + 2 \qquad b = \frac{3(2s'_n)^2}{2}.$$

For example:
If $n = 0$ then $r_0 + s'_0\sqrt{6} = \quad (3 + \sqrt{6})(5 + 2\sqrt{6})^0$ hence: $(r_0, s'_0) =\ (3, 1)$ and $a = b = 6$.
We have a balanced integral tree $T(1; 6|6)$.

If $n = 1$ then $r_1 + s'_1\sqrt{6} \quad = \quad (3 + \sqrt{6})(5 + 2\sqrt{6})$, hence $(r_1, s'_1) =\ (27, 11)$ and $a = 486,\ \ b = 726$.
We obtain a non balanced tree $T(1; 486|726)$, which has the smallest number of vertices from all trees of this class. The characteristics polynomial of its divisor is;

$$P_D = (x^4 - (1213)x^2 + 352836 \quad = (x^2 - 27^2)(x^2 - 22^2)$$

and its spectrum is $S_D = \{\pm 27, \pm 22\}$.

If $n = 2$ then $r_2 + s'_2\sqrt{6} \quad = \quad (2 + \sqrt{6})(3 + 2\sqrt{6})^2$, hence $(r_1, s'_1) =\ (267, 109)$ is the solution of its ordered pair and $a = 47526,\ \ b = 71286$.

Corollary 3. *Let* $s_1 = 3$. *Using the same method as in the Corollary 1*

$$4 = r^2 - 12s'^2.$$

Then

$$4 = r^2 - 12s'^2 \implies r^2 = 4 + 12s'^2.$$

Let $s' = 1$ then $(r, s') =\ (4, 1)$ hence; $4 + \sqrt{12}$ is the solution.
For

$$1 = r^2 - 12s'^2 \implies r^2 = 1 + 12s'^2.$$

Let $s' = 2$ then $(r, s') = (7, 2)$ hence; $7 + 2\sqrt{12}$ is the fundamental solution. Therefore all its solutions can be expressed by the formula;

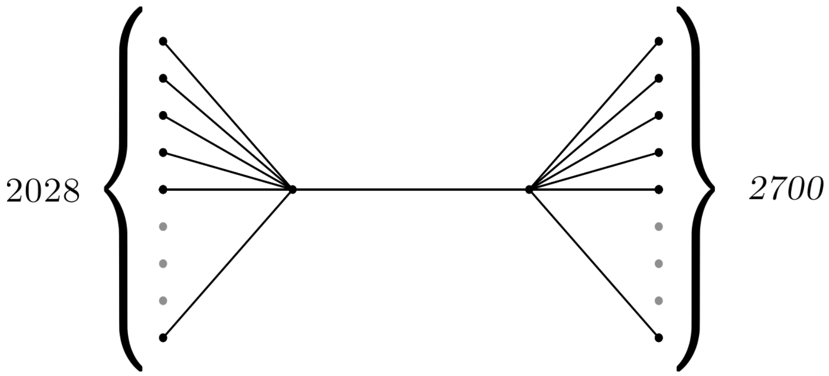
$$r_n + s'_n\sqrt{12} = (4 + \sqrt{12})(7 + 2\sqrt{12})^n.$$

where n = 0, 1, 2, 3, 4 , ...
From Theorem 3.2.1 above: The values of a and b are given as follows

$$a = (3s'_n)^2 + 3 \qquad b = \frac{4(3s'_n)^2}{3}.$$

For example:
If $n = 0$ then $r_0 + s'_0\sqrt{12} = (4 + \sqrt{12})(7 + 2\sqrt{12})^0$ hence: $(r_0, s'_0) = (4, 1)$ and $a = b = 12$.
We have a balanced integral tree $T(1; 12|12)$.

If $n = 1$ then $r_1 + s'_1\sqrt{12} = (4+\sqrt{12})(7+2\sqrt{12})$, hence $(r_1, s'_1) = (52, 15)$ and $a = 2028$, $b = 2700$.



We obtain a non balanced tree $T(1; 2028|2700)$, which has the smallest number of vertices from all trees of this class. The characteristics polynomial of its divisor is

$$P_D = x^4 - (4729)x^2 + 5475600 = (x^2 - 52^2)(x^2 - 45^2)$$

and its spectrum is $S_D = \{\pm 52, \pm 45\}$.

If $n = 2$ then $r_1 + s'_1\sqrt{12} = (4+\sqrt{12})(7+2\sqrt{12})^2$, hence $(r_1, s'_1) = (194, 56)$ and is the solution of its ordered pair and $a = 28227$, $b = 37632$.

Corollary 4. *Let $s_1 = 4$. Using the same method as in the Corollary 1*

$$5 = r^2 - 20s'^2.$$

Then

$$5 = r^2 - 20s'^2 \implies r^2 = 5 + 20s'^2.$$

Let $s' = 1$ then $(r, s') = (5, 1)$ hence; $5 + \sqrt{20}$ is the solution.
For

$$1 = r^2 - 20s'^2 \implies r^2 = 1 + 20s'^2.$$

Let $s' = 2$ then $(r, s') = (9, 2)$ hence; $9 + 2\sqrt{20}$ is the fundamental solution.
Therefore all its solutions can be expressed by the formula

$$r_n + s'_n\sqrt{20} \quad = \quad (5 + \sqrt{20})9 + 2\sqrt{20})^n$$

where n = 0, 1, 2, 3, 4 , ...
From Theorem 3.2.1 above: The values of a and b are given as follows

$$a = (4s'_n)^2 + 4 \qquad b = \frac{5(4s'_n)^2}{4}.$$

For example:
If $n = 0$ then $r_0 + s'_0\sqrt{12} = \quad (5 + \sqrt{20})(9 + 2\sqrt{20})^0$
;hence: $(r_0, s'_0) = (5, 1)$ and $a = b = 20$.
We have a balanced integral tree $T(1; 20|20)$.

If $n = 1$ then $r_1 + s'_1\sqrt{20} \quad = \quad (5+\sqrt{20})(9+2\sqrt{20})$, hence $(r_1, s'_1) = (85, 19)$ and $a = 5780$, $b = 7220$.
We obtain a non balanced tree $T(1; 5780|7220)$, which has the smallest number of vertices from all trees of this class. The characteristics polynomial of its divisor is

$$P_D = x^4 - (13001)x^2 + 41731600 \quad = (x^2 - 85^2)(x^2 - 76^2)$$

and its spectrum is $S_D = \{\pm 85, \pm 76\}$.

If $n = 2$ then $r_1 + s'_1\sqrt{20} \quad = \quad (5 + \sqrt{20})(9 + 2\sqrt{20})^2$, hence $(r_1, s'_1) = (1525, 341)$ and is the solution of its ordered pair and $a = 1860500$, $b = 2325620$.

Generally: It is proved that the problem of characterizing integral trees of diameter 3 is equivalent with the problem of solving Pell's Diophantine equations $x^2 - Dy^2 = L$ for appropriate integers.
Where;
$L := s_1 + 1$,
$D := (s_1 + 1)s_1$,
s_1: arbitrary any positive integer,

$x^2 = r^2, y^2 = (s')^2$.
The general solution of the Pell's Diophantine equations is given by:-

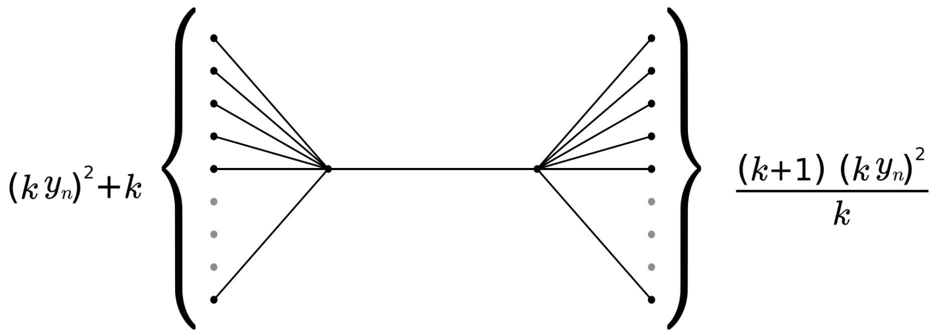
$$r_n + s'_n\sqrt{D} = (r_0 + s'_0\sqrt{D}) \underbrace{(r + s'\sqrt{D})^n}_{fundamental\ \ solution}.$$

The values of a and b are given by $s^2 + s_1$ and $\frac{(s_1+1)s^2}{s_1}$ respectively.
Where; $s = s_1 \cdot s'$.

Using Theorem 3.2.1 above and the succeeding corollaries; which generalize the Pell's equation and Integral trees $T(1; a|b)$ of diameter 3 from Pokorny in [16]. We develop the following theorem.

Theorem 3.2.2. *Let k be a given positive integer and $a = (ky_n)^2 + k$, $b = (k+1)(ky_n)^2/k$, $\forall n = 0, 1, 2, 3, \ldots$, then, $T(1; a|b)$ is an integral tree and its spectrum (S_D) is $\{\pm x_n, \ \pm y_n\}$.*

Proof. Consider the Pell's Diophantine equation $x^2 - k(k+1)y^2 = k+1$ and an integral tree $T(1; (ky_n)^2 + k \ | \ (k+1)(ky_n)^2/k)$.



Where $k(k+1)$ is not a perfect square.
Then; $(k+1, 1)$ is a solution, $(2k+1, 2)$ is the fundamental solution of this Diophantine equation and all positive integral solutions of $(x_n, \ y_n)$ are given by;

$$x_n + y_n\sqrt{k(k+1)} = (k+1+\sqrt{k(k+1)})(2k+1+2\sqrt{(k+1)k})^n \qquad \forall k = 1, 2, 3, \ldots.$$

We have;

$$x_n + y_n\sqrt{k(k+1)} = (k+1+\sqrt{k(k+1)})(2k+1+2\sqrt{k(k+1)})^n$$

$$x_n - y_n\sqrt{(k+1)k} = (k+1-\sqrt{k(k+1)})(2k+1-2\sqrt{k(k+1)})^n.$$

Let
$$A = (k+1+\sqrt{k(k+1)})(2k+1+2\sqrt{k(k+1)})^n$$

$$B = (k+1-\sqrt{k(k+1)})(2k+1-2\sqrt{k(k+1)})^n.$$

Then

$$x_n = \frac{A+B}{2}, \qquad y_n = \frac{A-B}{2\sqrt{k(k+1)}}.$$

□

Using this generalized integral solution we can generate finitely many integral trees of diameter 3. Theorem (3.2.2) also is useful in obtaining the spectrum of any integral tree of diameter 3.

For example:
If $k = 11$, then; $x^2 - 132y^2 = 12$. This Pell's equation give the following general solution

$$x_n + y_n\sqrt{132} \quad = \quad (12+\sqrt{132})(23+2\sqrt{132})^n.$$

By using the above Theorem (3.2.2) we can obtain finitely many integrals trees and their spectrum.

n	**a**	**b**
1	267300	291588
2	565060452	616429572
3	1194537517000	1303131836000
4	2525251745000000	2754820086000000
5	5338380994000000000	5823688358000000000

Table 3.1: This table shows some integral trees of diameter 3.

For $n = 1$: We obtain a non balanced tree $T(1; 267300|291588)$, which has the smallest number of vertices from all trees of this class. The characteristics polynomial of its divisor is

$$P_D = x^4 - (558889)x^2 + 7.79414724 \times 10^{10} \quad = \quad (x^2 - 540^2)(x^2 - 517^2)$$

and its spectrum is $S_D = \{\pm 540, \pm 517\}$.

Remark. For concrete k the Pell's Diophantine equation $x^2 - k(k+1)y^2 = k+1$ may have more than one solution and when we have more than one solution it implies that we have more than one family of integral trees.

For example: If k or $k+1$ is a square, then we may have another families of integral tress in some classes as it is shown in the table below (Table 3.2)
For $k = 3, 4, 5 \ldots, 40$.

k	***Family 1***	***Family 2***
3	(4, 1)	(14, 4)
4	(5, 1)	
5	(6, 1)	
6	(7, 1)	
7	(8, 1)	
8	(9, 1)	(51, 6)
9	(10, 1)	
10	(11, 1)	
11	(12, 1)	
12	(13, 1)	
13	(14, 1)	
14	(15, 1)	
15	(16, 1)	(124, 8)
16	(17, 1)	
17	(18, 1)	
18	(19, 1)	
19	(20, 1)	
20	(21, 1)	
21	(22, 1)	

k	***Family 1***	***Family 2***
22	(23, 1)	
23	(24, 2)	
24	(25, 1)	(245, 10)
25	(26, 1)	
26	(27, 1)	
27	(28, 1)	
28	(29, 1)	
29	(30, 1)	
30	(31, 1)	
31	(32, 1)	
32	(33, 1)	
33	(34, 1)	
34	(35, 1)	
35	(36, 1)	(426, 12)
36	(37, 1)	
37	(38, 1)	
38	(39, 1)	
39	(40, 1)	

Table 3.2: This table shows some integral trees of diameter 3

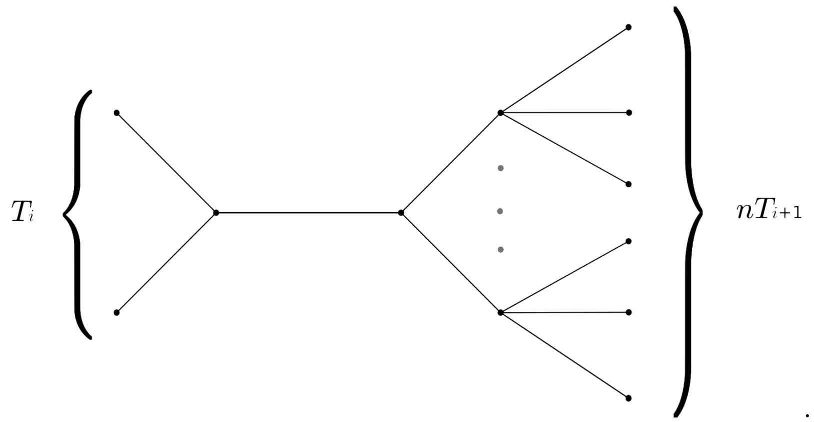
.

3.3 Integral Trees with even diameter

Many different classes of integral trees have been constructed in the past decades and most of these classes contain infinitely many integral trees [3]. Many researcher have characterized integral according to their diameter. In this section, we concentrate on trees constructed in [3] and compute their eigenvalues using a simple argument.

A rooted tree (T) is a tree with a specified vertex called the *root* [8]. We denote by F the forest resulting from removing the root of a tree. Let n be a positive integer and T_i, T_{i+1} be two rooted trees with disjoint vertex sets. Then, $T_i \sim nT_{i+1}$ is the rooted tree obtained from T_i and n copies of T_{i+1} by joining the root of T_i to the roots of the n copies of T_{i+1}.

For example: If we consider these two rooted trees below joining their root we obtain integral tree which has even diameter.



For positive integers $r_1 < r_2 < \ldots < r_n$, we define the rooted tree $Q(r_1, \ldots, r_n)$ [8]. For $n \geq 2$, with initial trees $Q()$ and $Q(r_1)$ being the one-vertex tree and the star tree on $r_1 + 1$ vertices, respectively we have;

$$Q(r_1, \ldots, r_n) = Q(r_1, \ldots, r_{n-2}) \sim (r_n - r_{n-1})Q(r_1, \ldots, r_{n-1}).$$

Lemma 3.3.1. *Let T_i, T_{i+1} be two rooted trees and let $T = T_i \sim nT_{i+1}$. Then;*

$$\varphi(T) = \varphi(T_{i+1})^{n-1}\varphi((T_i)\varphi(T_{i+1}) - n(\varphi(F_i)\varphi(F_{i+1})) \;\; [8].$$

This lemma is useful in determining the eigenvalues of trees constructed in [3] and their multiplicities.

Lemma 3.3.2. *Let $n \geq 2$ and $r_1, \ldots, r_n$ be positive integers. Then;*

$$(Q(r_1, \ldots, r_n)) \;=\; \varphi^{r_n - r_{n-1}}(Q(r_1, \ldots, r_{n-1}))(Q(r_1, \ldots, r_{n-2}))\frac{x^2 - r_n}{x^2 - r_{n-1}} \;\; [8].$$

Proof. Let

$$P_k = \varphi(Q(r_1, \ldots, r_k)) \quad C_k = \varphi(Q'(r_1, \ldots, r_k)) \quad d_k = r_k - r_{k-1}, \quad ,\forall k \geq 1, \quad r_0 = 0$$

Since

$$Q'(r_1, \ldots, r_k) = Q'(r_1, \ldots, r_{k-2}) \cup d_k(Q'(r_1, \ldots, r_{k-1})), \quad \forall k \geq 2.$$

Then we have $C_k = P_{k-1}^{d_k} C_{k-2}$. Using Lemma 3.3.1;

$$P_k = P_{k-1}^{d_{k-1}}(P_{k-1}P_{k-2} - (r_k - r_{k-1})C_{k-1}C_{k-2})$$

$$P_k = P_{k-1}^{d_{k-1}}(P_{k-2}^{d_k - 2}(P_{k-2}P_{k-3} - (r_{k-1} - r_{k-2})C_{k-2}C_{k-3}) - (r_k - r_{k-1})P_{k-2}^{d_{k-1}}C_{k-2}C_{k-3}))$$

$$P_k = P_{k-1}^{d_{k-1}}(P_{k-2}^{d_{k-2}}(P_{k-2}P_{k-3} - (r_k - r_{k-2})C_{k-2}C_{k-3})$$

$$\vdots$$

$$P_k = P_{k-1}^{d_{k-1}}(P_{k-1}P_{k-2}\dots P_2^{d_3})(P_2P_1 - (r_k - r_2)C_2C_1)$$

$$P_k = P_{k-1}^{d_{k-1}}(P_{k-1}P_{k-2}\dots P_2^{d_1})(P_1x - (r_k - r_1)C_1)$$

$$P_k = P_{k-1}^{d_{k-1}}(P_{k-1}P_{k-2}\dots P_2^{d_1})x - d_1(x^2 - r_k).$$

It is clear that $P_1 = x^{d_1} - (x^2 - r_1)$ and so $P_k = P_{k-1}^{d_k}(P_{k-2}^{d_k}\dots P_2^{d_1}x - d_1(x^2 - r_k)$ holds for $k \geq 1$. To complete the proof, apply this equality for $k = n-1, n$ and then compute P_n/P_{n-1}. $\square$

It is vivid that the diameter of $Q(r_1, \dots, r_n)$ is $2n$ provided that $r_n - r_{n-1} > 1$. The multiplicity of r_{n-1} as an eigenvalue of $Q(r_1, \dots, r_n)$ is $r_n - r_{n-1} - 1$. Using Lemma (3.3.2) lead to the theorem below which establishes the existence of infinitely many integral trees of any even diameter.

Theorem 3.3.1. *If $r_n - r_{n-1} > 1$, then the set of distinct eigenvalues of the tree $(Q(r_1, \dots, r_n))$ is $0, \pm\sqrt{r_1}\dots \pm \sqrt{r_n}$ [8].*

We introduce an alternative representation of $\varphi(Q(r_1, \dots, r_n))$ and $\varphi(Q'(r_1, \dots, r_n))$. Let

$$f(Q) = \prod_{i=1}^{\substack{\lceil 2 \rceil \\ n}} \frac{\varphi^{d_{n-2i+2}}(Q(r_1, \dots, r_{n-2i+2}))}{x^2 - 2i + 2}.$$

where $d_i = r_i - r_{i-1}$ with the convention $r_0 = 0$. From Lemma (3.3.2), $f(Q)$ is a polynomial and clearly we have;

$$\varphi(Q) = xf(Q)\prod_{i=1}^{\substack{\lceil 2 \rceil \\ n}}(x^2 - r_n - 2i + 2) \tag{3.3.1}$$

and

$$\varphi(Q)' = f(Q)\prod_{i=1}^{\substack{\lceil 2 \rceil \\ n}}(x^2 - r_n - 2i + 1). \tag{3.3.2}$$

Therefore, if $r_n - r_{n-1} > 1$ and $r_{n-1} - r_{n-2} > 1$, then the positive eigenvalues of $f(Q)$ read as $\sqrt{r_1}, \dots, \sqrt{r_{n-1}}$.

Theorem 3.3.2. *For every set S of positive integers there exists a tree whose positive eigenvalues are exactly the elements of S. If the set S is different from the set $\{1\}$ then the constructed tree will have diameter $2|S|$ [3].*

Proof. Trivially, there is only one tree with set S of positive eigenvalues for $S = 1$, and this is the tree on two vertices with spectrum {-1, 1} (and its diameter is 1).

Let $S = n_1, n_2, \ldots, n_{|S|}$ where $n_1 < n_2 < \ldots < n_{|S|}$. Then apply the previous theorem with $r_{|S|} = n_1^2, r_{|S|-1} = n_2^2 - n_1^2, \ldots, r_1 = n_{|S|}^2 - n_{|S|-1}^2$. If the set is different from $\{1\}$ then $r_1 \geq 2$ and in this case the diameter of the tree is $2|S|$. □

Example 1
Let $S = \{1, 2, 4, 5\}$ then $r_4 = 1, \quad r_3 = 3, \quad r_2 = 12, \quad r_1 = 9$. The resulting tree has 781 vertices and the spectrum is

$$\{-5, \quad -4^8, \quad -2^{100}, \quad -1^{227}, \quad 0^{109}, \quad 1^{227}, \quad 2^{100}, \quad 4^8, \quad 5\}.$$

Here the exponents are the multiplicities of the eigenvalues. The diameter of this tree is 8.

Example 2
Let $S = \{1, 2, 3, 4, 5, 6\}$ then $r_6 = 1, \quad r_5 = 3, \quad r_4 = 5, \quad r_3 = 7, \quad r_2 = 9, \quad r_1 = 11$. The resulting tree has 27007 vertices and the spectrum is;

$$\{\pm 6, \quad \pm 5^{10}, \quad \pm 4^{89}, \quad \pm 3^{611}, \quad \pm 2^{2944}, \quad \pm 1^{8021}, \quad 0^{3655}\}$$

The diameter of this tree is 12.

3.4 Integral graphs with odd diameter

In this section, we introduce a class of trees which will be used to obtain integral trees of odd diameters as discussed by Ghorbani, Mohammadian and Tayfeh-Rezaie in [8].

Theorem 3.4.1. *Let n be odd (respectively, even). Then, T is an integral tree of diameter $2n+1$ if and only if $r_0, r_1, \ldots, r_n$ are perfect squares and all the eigenvalues of $\varphi_0(T)$ (respectively) ,$\varphi_e(T)$ are integers [8].*

Proof. For positive integers $n, r, r_0, r_1, \ldots, r_n$ such that $n \geq 2$ and $\text{Max}\{r_0, r_1\} < r_2 < \ldots < r_n$. Let $U = Q(r_1, \ldots, r_n), V = Q(r_0, r_2, \ldots, r_{n-1}), W = Q(r_2, \ldots, r_n)$, and define

$$T(r, r_0, r_1, \ldots, r_n) \quad = \quad U \sim (V \sim rW).$$

We need to check the maximum distance between a vertex of $Q(k_1, \ldots, k_{n)}$ and its root is n. So, $T = T(r, r_0, r_1, \ldots, r_n)$ is a tree of diameter $2n+1$. We proceed to determine (T). Applying Lemma 3.3.1, we find that;

$$\varphi(T) = \varphi(U)\varphi^{r-1}(W)(\varphi(V)(W) - r\varphi(V)\varphi(W)) - \varphi(U')\varphi(V')\varphi^{r}(W)$$

=

$$\varphi^{r-1}(W)\varphi(U)\varphi(V)\varphi(W)) - r\varphi(U)\varphi(V')\varphi(W') - \varphi(U')\varphi(V')\varphi(W).$$

If $n = 2m+1$ odd, then using equation 3.3.1 and equation 3.3.2 we have:-

$$\varphi(U) = xf(U)(x^2 - r_1)(x^2 - r_n)\prod_{i=1}^{m}(x^2 - r_{2i-1}) \tag{3.4.1}$$

$$\varphi(V) = xf(V)\prod_{i=1}^{m}(x^2 - r_{2i-1}) \tag{3.4.2}$$

$$\varphi(W) = xf(W)(x^2 - r_n)\prod_{i=1}^{m}(x^2 - r_{2i-1}) \tag{3.4.3}$$

and

$$\varphi(U') = x^2 f(U)\prod_{i=1}^{m}(x^2 - r_{2i}) \tag{3.4.4}$$

$$\varphi(V') = f(V)(x^2 - r_0)\prod_{i=2}^{m}(x^2 - r_{2i-1}) \tag{3.4.5}$$

$$\varphi(W') = f(W)\prod_{i=1}^{m}(x^2 - r_{2i}). \tag{3.4.6}$$

Hence using equation 3.4.1 to equation 3.4.6we obtain:-

$$\varphi(T) = x(x^2 - r_n)\varphi^{r-1}(W)f(U)f(V)f(W)\prod_{i=2}^{m}(x^2 - r_n)\prod_{i=2}^{m}(x^2 - r_{2i-1})^2\varphi_0(T) \tag{3.4.7}$$

$$\varphi_0(T) = x^2(x^2 - r_1)(x^2 - r_n) - r(x^2 - r_0)(x^2 - r_1) - x^2(x^2 - r_0) \tag{3.4.8}$$

suppose that $n = 2m$ is even. Then using equation 3.3.1 and equation 3.3.2 we have:-

$$\varphi(U) = xf(U)(x^2 - r_n) \prod_{i=1}^{m-1} (x^2 - r_{2i}) \tag{3.4.9}$$

$$\varphi(V) = xf(V)(x^2 - r_0) \prod_{i=1}^{m} (x^2 - r_{2i-1}) \tag{3.4.10}$$

$$\varphi(W) = xf(W)(x^2 - r_n) \prod_{i=1}^{m-1} (x^2 - r_{2i}) \tag{3.4.11}$$

and

$$\varphi(U') = f(U)(x^2 - r_1) \prod_{i=2}^{m} (x^2 - r_{2i-1}) \tag{3.4.12}$$

$$\varphi(V') = x^2 f(V) \prod_{i=1}^{m-1} (x^2 - r_{2i}) \tag{3.4.13}$$

$$\varphi(W') = x^2 f(W) \prod_{i=2}^{m} (x^2 - r_{2i-1}). \tag{3.4.14}$$

Therefore using equation 3.4.9 to equation 3.4.14we obtain:-

$$\varphi(T) = x^3(x^2 - r_n)\varphi^{r-1}(W)f(U)f(V)f(W) \prod_{i=2}^{m} (x^2 - r_{2i-1}) \prod_{i=1}^{m-1} (x^2 - r_{2i-1})^2 \varphi_e(T) \tag{3.4.15}$$

$$\varphi_e(T) = (x^2 - r_0)(x^2 - r_n) - rx^2 - (x^2 - r_1).$$

□

Theorem 3.4.2. *For every even integer $n \geq 2$, there are infinitely many integral trees of diameter $2n + 1$.*

Proof. Let n be even. Then choose the parameters of $T = T(r, r_0, r_1, \ldots, r_n)$ in such a way that the eigenvalues of $\varphi_e(T)$ are all integers. For instance, let $r_0 = 1, \quad r_1 = 4k^2, \quad r_n = (k^2 - 1)^2$ and $r = 4k^2 - 1$. Then;

$$\varphi_e(T) = (x^2-1)(x^2-(k^2-1)^2)-(4k^2-1)x^2-(x^2-4k^2) = (x^2-1)(x^2-(k^2+1)^2).$$

It is trivial that if we choose k large enough, then we are able to take distinct prefect squares $r_2, \ldots, r_{n-1}$ in the interval $(4k^2, (k^2 - 1)^2)$. □

Theorem(3.4.2) lead to the formation of the class of integral trees of diameter $4k + 1$.

Theorem 3.4.3. *For every odd integer $n \geq 3$, there are infinitely many integral trees of diameter $2n + 1$.*

Proof. Let n be odd. Our aim is to choose the parameters of $T = T(r, r_0, r_1, \ldots, r_n)$ in such a way that all the eigenvalues of $\varphi_0(T)$ are integers. This can be done in many ways. For instance, if we set $r_0 = r_1 = a^2$ and $r = r_n = 4(a-1)^2$ for some integer a with $|a| \geq 3$,then; $\varphi_0(T) = (x^2 - a^2)(x^4 - (8(a-1)^2 + 1)x^2 + 4a^2(a-1)^2)$. The eigenvalues of $\varphi_0(T)$ are $\pm a$ and $\pm(a - \frac{3}{2}) \pm \frac{1}{2}\sqrt{12a^2 - 20a + 9}$. So, the eigenvalues of $\varphi_0(T)$ are integers if and only if $12a^2 - 20a + 9$ is a perfect square, say b^2. We have $(6a-5)^2 - 3b^2 = -2$. Then from Integral Trees and Pell's equation, we know that the Pell-like equation $x^2 - 3y^2 = -2$ has infinitely many integral solutions with $x \equiv \pm 1 \pmod 6$. For example, one may take;

$$a = \frac{1}{12}((1-\sqrt{3})(-2+\sqrt{3})k + (1+\sqrt{3})(-2-\sqrt{3})k + 10)$$

for arbitrary integer $k \geq 2$. Now, we are able to take the distinct prefect squares $r_2, \ldots, r_{n-1}$ in the interval $(a^2, 4(a-1)^2)$. $\square$

This theorem give a new class of integral trees of diameter $4k + 3$.

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