– Discrete Mathematics –

10th December, 2019

Name: .... Total: ....

Exercise 1. *Draw a Venn diagram for the following sets:*
$A \cup B \cup C = \{4, 6, 7, 9, 12, 14, 21, 35\}$
A contains even numbers,
B contains numbers divisible by 3,
C contains numbers divisible by 7. *(4 points)*

Exercise 2. *How many* 3*-element subsets does the set*

$$A = \{u + v : u \in \{-1, 0, 1, 2\}, v \in \{0, 1\}\}$$

have? *(4 points)*

Exercise 3. *Use the Euclidean algorithm to find* $\gcd(2291, 1363)$. *(4 points)*

Exercise 4. *Expand the following expression using the binomial theorem:*

$$\left(-\frac{x}{3} + 3\right)^3.$$

(4 points)

Exercise 5.
(a) How many eight digit numbers can be formed from the digits 1,1,1,2,4,3,3,4?

(b) How many seven digit numbers can be formed from the digits 2,3,1,2,3,0,4? *(4 points)*

Exercise 6. *Describe all values of n and k for which*

$$\binom{n}{k+1} = 4\binom{n}{k}.$$

(4 points)

Exercise 7. *a. Determine the decimal representation of the following numbers.*

$210_3 =$ .... $_{10}$

$210_4 =$ .... $_{10}$

$210_5 =$ .... $_{10}$ *(6 points)*

b. Determine the appropriate representations of the following numbers.

$2121_{10} =$ .... $_7$

$2121_{10} =$ .... $_8$

$2121_{10} =$ .... $_9$ *(6 points)*

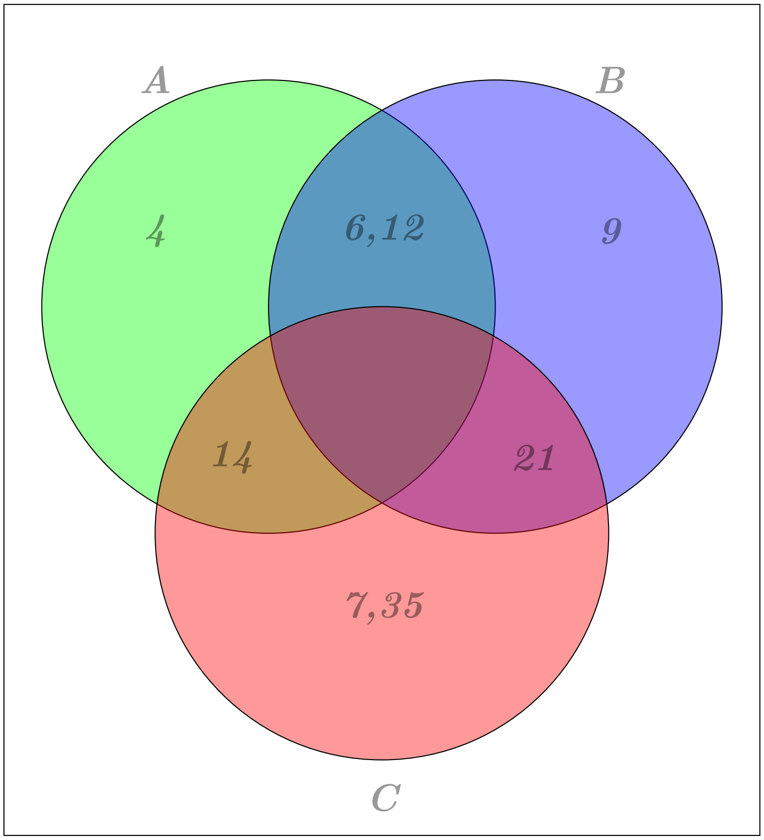
Exercise 8. *Prove by induction that*

$$\sum_{k=1}^{n} (3 - 2k) = n(2 - n)$$

for any positive integer n. *(8 points)*

Solutions

Solution 1.



Solution 2. *The set is given by* $A = \{-1, 0, 1, 2, 3\}$, *hence the number of 3-element subsets is* $\binom{5}{3}$.

Solution 3.

```
2291 = 1363 * 1 + 928
1363 = 928 * 1 + 435
928 = 435 * 2 + 58
435 = 58 * 7 + 29
58 = 29 * 2 + 0
```

Therefore $\gcd(2291, 1363) = 29$.

Solution 4.

$$\left(-\frac{x}{3}+3\right)^3 = \sum_{k=0}^{3} \binom{3}{k} \left(-\frac{x}{3}\right)^{4-k} 3^k = -\frac{1}{27}\,x^3 + x^2 - 9\,x + 27.$$

Solution 5.

(a)

$$\frac{8!}{3!1!2!2!},$$

(b)

$$\frac{7!}{1!1!2!2!1!} - \frac{6!}{1!2!2!1!}.$$

Solution 6. *The equation can be written as*

$$\frac{n!}{(n-k-1)!(k+1)!} = 4\frac{n!}{(n-k)!k!}.$$

Let us divide by $n!$ *and multiply by* $(n-k)!(k+1)!$. *We also make use of the recurrence definition of* $n!$, *that is* $n! = n \cdot (n-1)!$. *We get that*

$$\frac{(n-k)!}{(n-k-1)!} = 4\frac{(k+1)!}{k!}$$

that yields

$$n - k = 4k + 4 \longrightarrow n = 5k + 4.$$

Solution 7.

(a)

$$\begin{aligned} 210_3 &= 0 \cdot 3^0 + 1 \cdot 3^1 + 2 \cdot 3^2 = 0 + 3 + 18 = 21 \\ 210_4 &= 0 \cdot 4^0 + 2 \cdot 4^1 + 2 \cdot 4^2 = 0 + 4 + 32 = 36 \\ 210_5 &= 0 \cdot 5^0 + 2 \cdot 5^1 + 2 \cdot 5^2 = 0 + 5 + 50 = 55. \end{aligned}$$

(b)
```
2021 = 288 * 7 + 5
288 = 41 * 7 + 1
41 = 5 * 7 + 6
5 = 0 * 7 + 5
```
$2021_{10} = 5615_7$.
```
2021 = 252 * 8 + 5
252 = 31 * 8 + 4
31 = 3 * 8 + 7
3 = 0 * 8 + 3
```
$2021_{10} = 3745_8$.
```
2021 = 224 * 9 + 5
224 = 24 * 9 + 8
24 = 2 * 9 + 6
2 = 0 * 9 + 2
```
$2021_{10} = 2685_9$.

Solution 8.

1^{st} *Step: If* $n = 1$, *then* $\sum_{k=1}^{1}(3-2k) = 1$ *and* $1 \cdot (2-1) = 1$, *so the statement is true.*

2^{nd} *Step: Assume for some* m *that the statement is true, that is*

$$\sum_{k=1}^{m}(3-2k) = m(2-m).$$

3^{rd} *Step: We need to prove the statement for* $m+1$:

$$\sum_{k=1}^{m+1}(3-2k) = (m+1)(2-m-1) = (m+1)(1-m).$$

Let us rewrite the above sum:

$$\sum_{k=1}^{m+1}(3-2k) = \left(\sum_{k=1}^{m}(3-2k)\right) + (3-2(m+1)) = m(2-m) + (1-2m) = 1 - m^2 = (1+m)(1-m).$$

Hence the statement is true for $m+1$.