Discrete Mathematics

10th December, 2019

Name: .... Total: ....

Exercise 1. *Provide three sets A, B and C which satisfy the following cardinality conditions*

$$|A \cap B \cap C| = 0,$$
$$|A \cap B| = 2, \quad |A \cap C| = 2, \quad |B \cap C| = 2,$$
$$|A| = 5, \quad |B| = 6, \quad |C| = 7.$$

(4 points)

Exercise 2. *How many 2-element subsets does the set $A = \{u + v : u \in \{1, 3, 5\}, v \in \{2, 4\}\}$ have?* *(4 points)*

Exercise 3. *Use the Euclidean algorithm to find* $\gcd(2263, 1457)$. *(4 points)*

Exercise 4. *Expand the following expression using the binomial theorem:*

$$\left(-\frac{x}{2} + 2\right)^4.$$

(4 points)

Exercise 5. *How many solutions does the equation $x_1 + x_2 + x_3 + x_4 + x_5 = 25$ have, where x_1, x_2, x_3, x_4, x_5 are integers such that $x_k \geq (-1)^k + k$?* *(4 points)*

Exercise 6. *Describe all values of n and k for which*

$$\binom{n}{k+1} = 10\binom{n}{k}.$$

(4 points)

Exercise 7. *a. Determine the decimal representation of the following numbers.*

$120_5 =$ .... $_{10}$

$120_6 =$ .... $_{10}$

$120_7 =$ .... $_{10}$ *(6 points)*

b. Determine the appropriate representations of the following numbers.

$2019_{10} =$ .... $_4$

$2019_{10} =$ .... $_5$

$2019_{10} =$ .... $_7$ *(6 points)*

Exercise 8. *Prove by induction that 6 divides*

$$\sum_{k=0}^{2n-1} 5^k$$

for any positive integer n. *(8 points)*

Solutions

Solution 1.

$$\begin{aligned} A &= \{1, 2, 3, 4, 7\}, \\ B &= \{1, 2, 5, 6, 8, 9\}, \\ C &= \{3, 4, 5, 6, 10, 11, 12\}. \end{aligned}$$

Solution 2. *We have that* $A = \{3, 5, 7, 9\}$, *hence the number of 2-element subsets is* $\binom{4}{2}$.

Solution 3.

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2263 = 1457 * 1 + 806
1457 = 806 * 1 + 651
806 = 651 * 1 + 155
651 = 155 * 4 + 31
155 = 31 * 5 + 0
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Therefore $\gcd(2263, 1457) = 31$.

Solution 4.

$$\left(-\frac{x}{2} + 2\right)^4 = \sum_{k=0}^{4} \binom{4}{k} \left(-\frac{x}{2}\right)^{4-k} 2^k = \frac{1}{16} x^4 - x^3 + 6\,x^2 - 16\,x + 16.$$

Solution 5. *The system of inequalities is as follows*

$$\begin{aligned} x_1 &\geq (-1)^1 + 1 = 0 \to (x_1 + 1) \geq 1 \\ x_2 &\geq (-1)^2 + 2 = 3 \to (x_2 - 2) \geq 1 \\ x_3 &\geq (-1)^3 + 3 = 2 \to (x_3 - 1) \geq 1 \\ x_4 &\geq (-1)^4 + 4 = 5 \to (x_4 - 4) \geq 1 \\ x_5 &\geq (-1)^5 + 5 = 4 \to (x_5 - 3) \geq 1. \end{aligned}$$

We obtain that

$$(x_1 + 1) + (x_2 - 2) + (x_3 - 1) + (x_4 - 4) + (x_5 - 3) = 25 + 1 - 2 - 1 - 4 - 3 = 16.$$

Thus the number of solutions is $\binom{16-1}{5-1} = \binom{15}{4}$.

Solution 6. *The equation can be written as*

$$\frac{n!}{(n-k-1)!(k+1)!} = 10 \frac{n!}{(n-k)!k!}.$$

Let us divide by $n!$ *and multiply by* $(n-k)!(k+1)!$. *We also make use of the recurrence definition of* $n!$, *that is* $n! = n \cdot (n-1)!$. *We get that*

$$\frac{(n-k)!}{(n-k-1)!} = 10 \frac{(k+1)!}{k!}$$

that yields

$$n - k = 10k + 10 \longrightarrow n = 11k + 10.$$

Solution 7.

(a)

$$\begin{aligned} 120_5 &= 0 \cdot 5^0 + 2 \cdot 5^1 + 1 \cdot 5^2 = 0 + 10 + 25 = 35 \\ 120_6 &= 0 \cdot 6^0 + 2 \cdot 6^1 + 1 \cdot 6^2 = 0 + 12 + 36 = 48 \\ 120_7 &= 0 \cdot 7^0 + 2 \cdot 7^1 + 1 \cdot 7^2 = 0 + 14 + 49 = 63. \end{aligned}$$

(b)

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2019 = 504 * 4 + 3
504 = 126 * 4 + 0
126 = 31 * 4 + 2
31 = 7 * 4 + 3
7 = 1 * 4 + 3
1 = 0 * 4 + 1
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$2019_{10} = 133203_4$.

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2019 = 403 * 5 + 4
403 = 80 * 5 + 3
80 = 16 * 5 + 0
16 = 3 * 5 + 1
3 = 0 * 5 + 3
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$2019_{10} = 31034_5.$

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2019 = 288 * 7 + 3
288 = 41 * 7 + 1
41 = 5 * 7 + 6
5 = 0 * 7 + 5
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$2019_{10} = 5613_7.$

Solution 8.

1^{st} *Step: If* $n = 1$, *then* $\sum_{k=0}^{2\cdot 1-1} 5^k = 5^0 + 5^1 = 6$ *and it is divisible by 6.*

2^{nd} *Step: Assume for some* m *that the statement is true, that is*

$$\sum_{k=0}^{2m-1} 5^k = 6A$$

for some integer A.

3^{rd} *Step: We need to prove the statement for* $m+1$:

$$\sum_{k=0}^{2(m+1)-1} 5^k = \sum_{k=0}^{2m+1} 5^k$$

is divisible by 6. Let us rewrite the above sum:

$$\sum_{k=0}^{2m+1} 5^k = \left(\sum_{k=0}^{2m-1} 5^k \right) + 5^{2m} + 5^{2m+1} = 6A + 5^{2m}(1+5) = 6(A + 5^{2m}).$$

Hence the statement is true for $m+1$.