EQUAL VALUES OF FIGURATE NUMBERS

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ABSTRACT. Some effective results for the equal values of figurate numbers are proved. Using a state-of-the-art computational method for the small parameter values the corresponding Diophantine equations are resolved.

1. INTRODUCTION

There are several results concerning arithmetical and Diophantine properties of certain combinatorial numbers. Let k, m be integers with $k \geq 3$ and $m \geq 3$, further, denote by

$$f_{k,m}(X) = \frac{X(X+1)\dots(X+k-2)((m-2)X+k+2-m)}{k!}$$

the Xth figurate number with parameters k and m. For some problems and theorems related to these families of combinatorial numbers, we refer to the books [11] and [10]. The power and equal values of special cases of $f_{k,m}(X)$, including, for instance, binomial coefficients (for $m = 3$), polygonal numbers (for $k = 2$) and pyramidal numbers (for $k = 3$) have been studied intensively, see [1], [20], [4], [23], [8], [9], [14], [18], [19], [17], [16] and references therein. Brindza, Pintér and Turjányi [5] conjectured that apart from the case $(m, n) = (5, 4)$ the equation

$$f_{3,m}(x) = f_{2,n}(y)$$

has only finitely many solutions in integers x, y which can be effectively determined. Recently, Pintér and Varga [24] confirmed this conjecture.

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The purpose of the note at hand is to give effective finiteness statements for the more general equation

$$f_{k,m}(x) = f_{2,n}(y) \tag{1}$$

in integers x and y and to provide numerical results for small values of parameters (k, m, n). In a forthcoming paper we will deal with the equation

$$f_{k,m}(x) = f_{l,n}(y).$$

However, in this generality we can give only ineffective finiteness theorems.

2. Main results

Theorem 2.1. *Let m, n, k be integers with $k \geq 3$ and $(m, n, k) \neq (5, 4, 3), (6, 4, 4)$. If k is even, then assume further that $k!D$ is not of the form $r^2, 2r^2$, where $D = \gcd(k!(n-4)^2, 8d(n-2))$ with $d = \gcd(k, m-2)$. Then equation (1) has only finitely many solutions in x, y which can be effectively determined.*

If $(m, n, k) = (5, 4, 3), (6, 4, 4)$, then one can easily see that equation (1) has infinitely many solutions in x, y. As an immediate consequence, we obtain the following statement.

Corollary 2.1. *Let m, n, k be integers with $k \geq 4$. If k is even, then assume further that there exists a prime p with $k/2 < p < k$ such that $p \nmid n-2$. Then equation (1) has only finitely many solutions in x, y which can be effectively determined.*

Remark. Note that if $k > 2n$, then the condition in Corollary 2.1 is satisfied. Indeed, Bertrand's postulate guarantees the existence of a prime p with $k/2 < p < k$. Since now $p > k/2 = n > n-2$, we also have $p \nmid n-2$.

Theorem 2.2. *Suppose that $k \geq 3, m \geq 3, n \geq 14$ are integers with*

$$10m - 26 \leq n.$$

Then equation (1) possesses only finitely many solutions in x, y which can be effectively determined.

We closely follow arguments of Erdős [12, 13] and resolve an infinite family of Diophantine equations.

Theorem 2.3. *The only solution of the equation*

$$f_{k,k+2}(x) = f_{2,4}(y) \tag{2}$$

in integers $k \geq 5$, $x \geq k-2$ and $y \geq 1$ is $(k, x, y) = (5, 47, 3290)$.

For $k = 5$, our theorem follows from a classical theorem by Meyl [22]. The resolution of another parametric family of Diophantine problems

$$\binom{x+k-1}{k} = f_{k,3}(x) = f_{2,4}(y) = y^2$$

in integers x, y and k follows from the result of Győry [14] on the power values of binomial coefficients.

Consider now the case $k = 5$. Then equation (1) can be reduced to the Diophantine equation

$$(3)\quad 15(n-2)x(x+1)(x+2)(x+3)((m-2)x+7-m)+(15(4-n))^2 = z^2,$$

where $z = 30(n-2)y + 15(4-n)$.

The curve (3) is a genus 2 hyperelliptic curve except for finitely many pairs of (m, n), where $m, n \geq 3$. The exceptional pairs (m, n) could be explicitly given by Runge's method. However, this would require a lot of calculations, involving a large amount of technical data. Since this point is not vital for our purposes, we suppress the details.

We computed the rank r (an upper bound for the rank in some cases) of the Jacobian of the corresponding hyperelliptic curve for $m, n \in \{3, 4, 5, 6, 7, 8\}$.

$n \setminus m$	3	4	5	6	7	8
3	6	5	5	6	4	6
4	$1 \leq r \leq 5$	$2 \leq r \leq 6$	$2 \leq r \leq 6$	$3 \leq r \leq 7$	-	$1 \leq r \leq 5$
5	4	5	4	4	2	5
6	6	5	5	6	4	6
7	5	5	5	5	4	5
8	6	5	7	7	4	6

We note that the problem in case of $(m, n) = (3, 3)$ yields the equation

$$\binom{x+4}{5} = \binom{y+1}{2}.$$

All integral points were determined by Bugeaud, Mignotte, Siksek, Stoll and Tengely [6] on the related curve hyperelliptic curve. They combined Baker's method and the so-called Mordell-Weil sieve to obtain the result. We follow their method to find all integral points on the curve (3) with $m = 7$ and $n = 5$, and hence to obtain all solutions of (1) for these values of parameters.

Theorem 2.4. *The set of integral points* (x, y) *on the curve* (3) *with* $(m, n) = (7, 5)$ *is*

$$\{(-3, 0), (-2, 0), (-1, 0), (0, 0), (1, 1)\}.$$

3. Auxiliary results

In the proof of Theorem 2.1 the next result plays the key role. In fact it provides more information than is needed to prove Theorem 2.1.

Proposition 3.1. *Let $t \geq 0$ be an integer, and write $P_t(x) = x(x+1)\dots(x+t)$. Let $f(x) \in \mathbb{Z}[x]$ and $v \in \mathbb{Z} \setminus \{0\}$ such that $g(x) := P_t(x)f(x) + v$ is a primitive polynomial.*

- *If $t \geq 3$ and $\deg(g)$ is odd, then $g(x)$ has at least three roots of odd multiplicities.*
- *If $t \geq 2$, $\deg(g)$ is even and v is not of the form $\pm r^2$, $\pm 2r^2$, then $g(x)$ has at least three roots of odd multiplicities.*
- *Let $\ell \geq 3$. If $t \geq 3$ and $\deg(f) < (t+1)(\ell-1)$, then $g(x)$ has at least two roots with multiplicities not divisible by ℓ.*

Proof. To prove the first part, suppose that $\deg(g)$ is odd, but it has less than three roots of odd multiplicities. Then we can write

$$P_t(x)f(x) + v = (h(x))^2(ax+b)$$

with some $h \in \mathbb{Z}[x]$ and $a, b \in \mathbb{Z}$. Further, $a \neq 0$, and by the primitivity of g we have $\gcd(a,b) = 1$. As $0, -1, -2, -3$ are roots of $P_t(x)$, we obtain

$$(h(0))^2 b = (h(-1))^2(b-a) = (h(-2))^2(b-2a) = (h(-3))^2(b-3a).$$

Observe that since $v \neq 0$, none of the above numbers is zero. As $\gcd(a,b) = 1$, this implies that either $b, b-a, b-2a, b-3a$ or $-b, a-b, 2a-b, 3a-b$ are all squares. However, by classical results of Euler and Fermat we have that four distinct squares cannot form an arithmetic progression (see [11], pp. 440 and 635). Hence our statement follows in this case.

To prove the second part, assume that $\deg(g)$ is even, but it has less than three roots of odd multiplicities. As v is not a square, by our assumptions $g(x)$ cannot be a constant (integral) multiple of a square of a polynomial in $\mathbb{Z}[x]$. Thus the only possibility is that we have

$$P_t(x)f(x) + v = (h(x))^2(ax^2 + bx + c)$$

with some $h \in \mathbb{Z}[x]$ and $a, b, c \in \mathbb{Z}$. Further, $a \neq 0$, and by the primitivity of g we have $\gcd(a,b,c) = 1$. Since $t \geq 2$, now we obtain

$$(h(0))^2 c = (h(-1))^2(a-b+c) = (h(-2))^2(4a-2b+c) = v.$$

As $v \neq 0$, none of the above numbers is zero. By a simple calculation we get that only $\gcd(c, a-b+c, 4a-2b+c) = 1, 2$ are possible. Assume that there is an odd prime q occurring on an odd power in the prime factorization of c. Then by the above equalities, q also occurs on an odd exponent in the prime factorization of v, whence $q \mid a-b+c$ and $q \mid 4a-2b+c$ follows. However, this is impossible. Hence c is one of the form $\pm r^2$, $\pm 2r^2$. But then the same is true for v, which is a contradiction. Hence the statement follows also in this case.

To prove the third part, suppose to the contrary that $g(x)$ has at most one root of multiplicity not divisible by ℓ. Consider first the case where $g(x)$ is an ℓ-th power in $\mathbb{Z}[x]$, that is

$$P_t(x)f(x)+v=(h(x))^\ell$$

with some $h \in \mathbb{Z}[x]$. Writing F and H for the degrees of f and h respectively, we get

$$t+1+F=\ell H.$$

On the other hand, by our assumption we have

$$F<(t+1)(\ell-1).$$

Combining these assertions, we obtain that $H<t+1$. On the other hand, we have

$$h(0)=h(-1)=\cdots=h(-t)=v,$$

that is, h takes the same value at $t+1$ different places. It yields that $h(x)$ is identically constant. It is a contradiction, and our statement follows in this case.

Finally, we are left with the possibility

$$P_t(x)f(x)+v=(h(x))^\ell(ax+b)^s$$

with some $h \in \mathbb{Z}[x]$, $a,b \in \mathbb{Z}$ with $\gcd(a,b)=1$ and s with $1 \le s < \ell$. As $t \ge 3$, we have

$$(h(0))^\ell b^s=(h(-1))^\ell(b-a)^s=(h(-2))^\ell(b-2a)^s=(h(-3))^\ell(b-3a)^s.$$

As $\gcd(a,b)=1$, similarly as in case of $\ell=2$ we get that

$$b^s,(b-a)^s,(b-2a)^s,(b-3a)^s$$

are all non-zero perfect ℓ-th powers. This by $s<\ell$ yields that

$$b,b-a,b-2a,b-3a$$

are all perfect ℓ'-th powers with some $\ell'=\frac{\ell}{gcd(s,\ell)} \ge 2$. However, by a deep result of Darmon and Merel [7] four distinct ℓ'-th powers cannot form an arithmetic progression. Hence our statement follows. □

Our next lemma is a classical result from the modern theory of Diophantine equations.

Lemma 3.1. *Let $t(X) \in \mathbb{Q}[X]$ and suppose that the polynomial $t(X)$ possesses at least three zeros of odd multiplicities. Then the equation $t(x) = y^2$ in integers x, y implies that $\max(|x|, |y|) < C$, where C is an effectively computable constant depending only on the polynomial $t(X)$.*

Proof. The result is a consequence of the Theorem in Brindza [3]. □

4. Proofs

Proof of Theorem 2.1. Equation (1) can be rewritten as

$$\frac{8(n-2)x(x+1)\dots(x+k-2)((m-2)x+k+2-m)}{k!} + (n-4)^2 = \\ = (2(n-2)y+n-4)^2. \tag{4}$$

So to prove the statement we only need to show that the polynomial $T(x)$ on the left hand side of the above equation has at least three zeroes of odd multiplicities. If $n = 4$, then one can easily check that this assertion is valid, provided that $(m, k) \neq (5, 3), (6, 4)$. So from this point on we may assume that $n \neq 4$.

Write $d := \gcd(k, m-2)$, and $D := \gcd(k!(n-4)^2, 8(n-2)d)$. Then we obviously have that $k!T(x)/D$ is a primitive polynomial in $\mathbb{Z}[x]$, with constant term $k!(n-4)^2/D$. Hence in view of Proposition 3.1, the theorem follows. □

Proof of Corollary 2.1. Observing that by $d \mid k$, we have $d = k$ or $d \leq k/2$. Further, $k \geq 4$ also yields $2 \leq k/2$. Hence if there exists a prime p with the desired properties, then obviously, p divides $k!(n-4)^2$ on an odd exponent, but $p \nmid D$ is valid. Thus the statement immediately follows from Theorem 2.1. □

Proof of Theorem 2.2. Observe that equation (1) can be rewritten as

$$8(n-2)f_{k,m}(X) + (n-4)^2 = z^2,$$

where $z = 2(n-2)y + n - 4$. Suppose that α is a multiple zero of the polynomial

$$8(n-2)f_{k,m}(X) + (n-4)^2 =$$
$$= \frac{8(n-2)(m-2)}{k!} X(X+1)(X+2)\dots\left(X + \frac{k}{m-2} - 1\right) + (n-4)^2.$$

Then α is a zero of the polynomial

$$g(X) := \left(X(X+1)\dots(X+k-2)\left(X + \frac{k}{m-2} - 1\right)\right)'.$$

In case of $1-\frac{k}{m-2} \notin H := \{0, -1, \dots, -k+2\}$, using Rolle's theorem one can check that these zeros are real and belong to the interval $(1-k, 1)$. When $1-\frac{k}{m-2} \in H$, this property can be easily verified by checking the sign of $g(X)$ in small neighborhoods of the elements of H. For $m = 3$ the statetement follows from a nice result of Yuan [28]. Thus we may assume that $m \geq 4$. Hence for an arbitrary real number $\beta \in (1-k, 1)$ the product

$$\left|\beta \cdot (\beta+1) \cdot \ldots \cdot (\beta+k-2)\left(\beta+\frac{k}{m-2}-1\right)\right|$$

is smaller than

$$(k-1)!\left(k-1-\frac{k}{m-2}+1\right) = k!\frac{m-3}{m-2},$$

that is

$$(n-4)^2 = |8(n-2)f_{k,m}(\alpha)| < 8(m-2)(n-2)\frac{m-3}{m-2} = 8(m-3)(n-2)$$

and this contradiction proves that there is no multiple zero of the polynomial

$$8(n-2)f_{k,m}(X) + (n-4)^2.$$

Observe that this inequality cannot hold for $n < 14$. As $10m - 26 \leq n$ implies that $8(m-3)(n-2) \leq (n-4)^2$ whenever $n \geq 14$ (and the statement is empty for $n < 14$), Lemma 3.1 finishes our proof. □

Proof of Theorem 2.3. Equation (2) can be rewritten as

$$x^2(x+1)\dots(x+k-2) = (k-1)!y^2. \tag{5}$$

First, using standard arguments, but with a slight modification implied by the presence of the factor $k-1$ on the right hand side of (5), we can write

$$x+i = a_i x_i^2 \quad (i = 1, \dots, k-2), \tag{6}$$

where the a_i are square-free positive integers with $P(a_i) \leq k-1$, where $P(u)$ denotes the greatest prime factor of u, with the convention $P(1) = 1$. First we prove that the coefficients a_i are pairwise different. Assume to the contrary that $a_i = a_j$ holds with some $i < j$. Then we have

$$k-2 > (x+j)-(x+i) = a_i x_j^2 - a_i x_i^2 = a_i(x_j^2 - x_i^2) \geq$$

$$\geq a_i((x_i+1)^2 - x_i^2) > 2\sqrt{a_i x_i^2} \geq 2\sqrt{x+1}.$$

On the other hand, as $x \geq k-2$, by Corollary 1 of Laishram and Shorey [21] we obtain that up to fourteen exceptions listed explicitly, the product $(x+1)\dots(x+k-2)$ has a prime factor $> 1.8(k-2)$. As one can easily check, these exceptions do not yield solutions to equation (2). Indeed, for example when $x+1=8$, $k-2=3$, the product is given by $8\cdot 9\cdot 10$, with greatest prime factor 5, and $5<1.8\cdot 3$. However, then we have $x=7$ and $k=5$, and equation (2) does not hold. The remaining exceptional case can be excluded similarly. Thus we may assume that q is a prime such that q divides $(x+1)\dots(x+k-2)$ and $q>1.8(k-2)$. Observe that then q divides exactly one term $x+i$ $(i=1,\dots,k-2)$. Since $q>k-1$ as $k\geq 5$, q occurs in $x+i$ on at least the second power. This yields

$$3.24(k-2)^2<q^2\leq x+k-2.$$

Combining this bound with the above estimate $k-2>2\sqrt{x+1}$, in view of $x\geq k-2$, we get a contradiction. This implies that $a_i\neq a_j$ indeed, whenever $i\neq j$.

Now we prove that the product $a_1\cdots a_{k-2}$ divides $(k-1)!$. For this, rewrite (2) as

$$A:=\frac{a_1\cdots a_{k-2}}{(k-1)!}=\frac{y^2}{z^2}$$

where $z=x\cdot x_1\cdots x_{k-2}$. Let p be any prime, and let $\nu_p(A)=\alpha$. Here $\nu_p(A)$ is the exponent of p in A; note that α may be negative, too. Then, recalling that the coefficients a_i are square-free, by Liouville's formula concerning the exponents of primes in a factorial we clearly have

$$\alpha\leq\left[\frac{k-2}{p}\right]+1-\left[\frac{k-1}{p}\right]\leq 1.$$

Since α must obviously be even, this yields $\alpha\leq 0$, which immediately implies our claim $a_1\cdots a_{k-2}\mid(k-1)!$. This of course gives

$$a_1\cdots a_{k-2}\leq(k-1)!.$$

If $5\leq k<15$, then the only solution is given by $(k,x,y)=(5,47,3290)$. This fact can be checked in the following way. First observe that by (6) we have

$$(x+1)\cdots(x+k-2)=uv^2, \tag{7}$$

where $u=a_1\cdots a_{k-2}$ and $v=x_1\cdots x_{k-2}$. Further, here $q\mid v$, therefore the greatest prime divisor of v is greater than $k-2$. Thus, by a result of Győry [15] we have that if $k-1$ is not a prime, then the only solution of (7) is given by $(x,k,u,v)=(47,5,6,140)$. This shows that the only solution to equation (2) is $(k,x,y)=(5,47,3290)$ in this case. Hence we may assume that $k-1$ is a prime, i.e. $k=6,8,12,14$. The investigation of these cases is similar, so we illustrate our method only for $k=6$. Then equation (2) is given by $x^2(x+1)(x+2)(x+3)(x+4)=120y^2$.

Checking the greatest common divisors of $x+i$ and $x+j$ ($1 \leq i < j \leq 4$), we get the possible values of a_1, a_2, a_3 in (6). Hence we obtain elliptic equations of the form

$$(x+1)(x+2)(x+3) = Az^2,$$

where A is square-free, and z is given by $z = Bx_1x_2x_3$ with $AB^2 = a_1a_2a_3$. It turns out that we have $A \in \{1, 2, 3, 5, 6, 10, 15, 30\}$. We used a MAGMA [2] code to solve these equations and we got that the only solutions are given by $(x, z) = (7, 12)$ with $A = 5$, $(x, z) = (0, 1), (1, 2), (47, 140)$ with $A = 6$, $(x, z) = (2, 2)$ with $A = 15$ and $(x, z) = (3, 2)$ with $A = 30$. shows that from these solutions one gets the only solution $(k, x, y) = (5, 47, 3290)$ for the original equation (2). It follows that the only solution of (2) is $(k, x, y) = (5, 47, 3290)$.

Assume now that $k \geq 15$. Then, since the numbers $a_1, \ldots, a_{k-2}$ are $k-2$ pairwise different square-free integers, we have

$$a_1 \ldots a_{k-2} \geq 1\cdot2\cdot3\cdot5\cdot6\cdot7\cdot10\cdot11\cdot13\cdot\ldots\cdot(k-1)\cdot k\cdot(k+1)\cdot(k+2) > (k-1)!.$$

(The second inequality follows from the fact that $k \cdot (k+1) \cdot (k+2) > 4 \cdot 8 \cdot 9 \cdot 12$, as $k \geq 15$.) However, this by the previous inequality yields a contradiction. That is, equation (2) has no solutions for $k \geq 15$, and the theorem follows. □

Proof of Theorem 2.4. The curve (3) with $m = 7$ and $n = 5$ is isomorphic to

$$X^2(X+1)(X+2)(X+3) + 1 = Y^2. \tag{8}$$

Let $J(\mathbb{Q})$ be the Jacobian of the genus two curve (8). Using MAGMA [2] we get that $J(\mathbb{Q})$ is free of rank 2 with Mordell-Weil basis given by

$$D_1 = (-1, 1) - \infty,$$
$$D_2 = (\omega, 2\omega + 3) + (\overline{\omega}, 2\overline{\omega} + 3) - 2\infty,$$

where ω is a root of the polynomial z^2+3z+2. The MAGMA procedures used to compute these data are based on Stoll's papers [25], [26], [27]. Let $f = X^2(X+1)(X+2)(X+3) + 1$ and α be a root of f. We will choose for coset representatives of $J(\mathbb{Q})/2J(\mathbb{Q})$ the linear combinations $\sum_{i=1}^{2} n_iD_i$, where $n_i \in \{0, 1\}$. Then

$$X - \alpha = \kappa\xi^2,$$

where κ is from a finite set. Such a finite set can be constructed following Lemma 3.1 in [6]. In case of the curve (8) we obtain that $\kappa \in \{1, -\alpha - 1, \alpha^2 + \alpha, \alpha^2 + 3\alpha + 2\}$. We applied Theorem 9.2 in [6] to get a large upper bound for $\log |X|$. A MAGMA code were written to obtain such bounds, it can be found at `http://www.warwick.ac.uk/~maseap/progs/intpoint/bounds.m`. In our case this bound turned out to be

$$\log|X| \leq 6.647 \times 10^{412}.$$

A search reveals 13 rational points on the genus 2 curve (8):

$$\infty, (-3, \pm 1), (-2, \pm 1), (-1, \pm 1),$$
$$(-7/4, \pm 17/32), (0, \pm 1), (1, \pm 5).$$

Let W be the image of the set of these known rational points in $J(\mathbb{Q})$. There are three points in the coset represented by 0:

$$\pm 6D_1 = (-7/4, \pm 17/32) - \infty$$

and ∞. There are two points in the same coset as D_1 :

$$\pm D_1 = (-1, \pm 1) - \infty.$$

In the coset of D_2 we obtain 6 points:

$$\pm(2D_1 + 3D_2) = (-3, \pm 1) - \infty,$$
$$\pm(2D_1 + D_2) = (0, \mp 1) - \infty,$$
$$\pm(2D_1 - 3D_2) = (1, \pm 5) - \infty.$$

Finally, two points belong to the coset of $D_1 + D_2$:

$$\pm(D_1 - D_2) = (-2, \pm 1) - \infty.$$

Applying the Mordell-Weil sieve explained in [6] we obtain that $\jmath(C(\mathbb{Q})) \subseteq W + BJ(\mathbb{Q})$, where

$$B = 2841720553897526432308772658708262465848000.$$

We follow an extension of the Mordell-Weil sieve due to Siksek to obtain a long decreasing sequence of lattices in $\mathbb{Z}^2$. After that we apply Lemma 12.1 in [6] to obtain a lower bound for possible unknown rational points. We have that if (X, Y) is an unknown integral point, then

$$\log|X| \geq 3.32 \times 10^{494}.$$

This contradicts the bound for $\log|X|$ obtained by Baker's method. Hence the set of integral points on the curve (8) is

$$\{(-3, \pm 1), (-2, \pm 1), (-1, \pm 1), (0, \pm 1), (1, \pm 5)\}.$$

These points correspond to the following set of integral points on (3):

$$\{(-3, 0), (-2, 0), (-1, 0), (0, 0), (1, 1)\}.$$

□

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